Homework 7 Selected Solutions
Due: Tuesday, March 17

3A.1 Suppose T is linear. First observe that $T(0,0,0) = (b,0)$. Since a linear map must send 0 to 0, we get that if T is linear, then $b = 0$. Next, observe that if T is linear, since $b = 0$

$$T(1,1,1) = (1, c+6),$$

while

$$T(2,2,2) = (2, 8c+12).$$

But we also have $T(2,2,2) = 2T(1,1,1)$, so $8c + 12 = 2(c+6)$. It follows that $8c = 2c$, and hence $c = 0$.

Now suppose $b = c = 0$. Then $T(x,y,z) = (2x - 4y + 3z, 6x)$. But we showed in class that maps of this form are in fact linear.

3A.3 Let $e_1, e_2, \ldots, e_n$ be the standard basis vectors for $\mathbb{F}^n$. Let $w_k = Te_k$ for $k = 1, \ldots, n$. As $w_k \in \mathbb{F}^m$, we have that for each k, $\exists A_{1,k}, \ldots, A_{m,k}$ such that

$$w_k = (A_{1,k}, A_{2,k}, \ldots, A_{m,k}).$$

I claim that the $A_{j,k}$ satisfy the claim. This holds because, as T is linear,

$$\begin{aligned} T(x_1, \ldots, x_n) &= T(x_1 e_1 + \cdots + x_n e_n) \\ &= x_1 T(e_1) + \cdots + x_n T(e_n) \\ &= x_1 w_1 + \cdots + x_n w_n \\ &= x_1(A_{1,1}, A_{2,1}, \ldots, A_{m,1}) + \cdots + x_n(A_{1,n}, A_{2,n}, \ldots, A_{m,n}) \\ &= (x_1 A_{1,1} + \cdots + x_n A_{1,n}, x_1 A_{2,1} + \cdots + x_n A_{2,n}, \ldots, x_1 A_{m,1} + \cdots + x_n A_{m,n}). \end{aligned}$$

The claim follows.

3A.7 First we find the scalar. Let v_1 be a basis for V. Let $w = T(v_1)$. Since $w \in V = \text{Span}(v_1)$, $\exists \lambda \in \mathbb{F}$ such that $w = \lambda v_1$.

Now I prove the claim, that $T(v) = \lambda v$ for all $v \in V$. So let $v \in V$. Since $V = \text{Span}(v_1)$, $\exists a \in \mathbb{F}$ such that $v = a v_1$. Then

$$\begin{aligned} T(v) &= T(a v_1) \\ &= a T(v_1) \text{ by linearity} \\ &= a w \\ &= a \lambda v_1 \\ &= \lambda a v_1 \\ &= \lambda v. \end{aligned}$$

The claim follows.