# Homework 5 Selected Solutions
## Due: Tuesday, February 24

2A.1 Let $U$ be the given subspace. I claim that

$$(1,0,-1),(0,1,-1),(1,-1,0),(2,-1,-1)$$

span $U$. I leave to the reader to check that these 4 lie in $U$. Thus by Prop 2.6, the span is a subset of $U$. Next, let $u \in U$. Write $u = (x,y,z)$. Since $x+y+z=0$, we have $z=-x-y$. Then

$$x(1,0,-1)+y(0,1,-1)=(x,y,-x-y)=u.$$

Thus $u$ lies in the span of the 4 vectors. (We only used the first two vectors, but we are not required to use all 4.) This shows that $U$ is a subset of the span, and so we have equality.

2A.12 If $v_1+w, v_2+w, \ldots, v_m+w$ is linearly dependent, $\exists a_1, \ldots, a_m \in \mathbb{F}$, not all zero, such that

$$a_1(v_1+w)+a_2(v_2+w)+\cdots+a_m(v_m+w)=0.$$

Rearranging, we get

$$a_1v_1+a_2v_2+\cdots+a_mv_m=(-a_1-a_2-\cdots-a_m)w. \tag{1}$$

Let $\lambda=-a_1-a_2-\cdots-a_m$. We would like to divide by $\lambda$, but to do that, we need to know that $\lambda \neq 0$. So suppose for the sake of contradiction that $\lambda=0$. Then since $0 \cdot w = 0$,

$$a_1v_1+\cdots+a_mv_m=0.$$

But $v_1,\ldots,v_m$ is linearly independent, so this would imply that the $a_i$ are all zero. Our earlier assumption was that they are *not* all zero! This is a contradiction. Therefore $\lambda \neq 0$.

Going back to equation (1), we divide both sides by $\lambda$ to obtain

$$\frac{a_1}{\lambda}v_1+\frac{a_2}{\lambda}v_2+\cdots+\frac{a_m}{\lambda}v_m=w,$$

which proves that $w \in \operatorname{Span}(v_1,\ldots,v_m)$.

2A.18 We will use Prop 2.22 and come up with arbitrarily long linearly independent lists. Let $e_i$ be the vector with a 1 in the $i$th place and 0 in all other places. For each $n$, I claim that $e_1,e_2,\ldots,e_n$ is linearly independent. (Note that we cannot "let $n=\infty$" since the definition of linear independence only applies to finite lists.) For given scalars $a_1,\ldots,a_n$,

$$a_1e_1+a_2e_2+\cdots+a_ne_n=(a_1,a_2,\ldots,a_n,0,0,0,\ldots).$$

Thus if the above equals the zero vector, then $a_i = 0$ for all $i$. This proves linear independence.

If $\mathbb{F}^\infty$ were finite-dimensional, there would be a finite spanning list, say of size $m$. Let $n = m + 1$. Then as shown above, there is a list of $m + 1$ linearly independent vectors. This contradicts Prop 2.22. Therefore $\mathbb{F}^\infty$ is infinite-dimensional.