Homework 4 Selected Solutions
Due: Tuesday, February 17

1C.19 It is false. Let $V = V_1 = U = \mathbb{R}^2$, and let $V_2 = \{0\}$. Certainly

$$V_1 + U = \mathbb{R}^2 + \mathbb{R}^2 = \mathbb{R}^2,$$

since $\mathbb{R}^2$ is the smallest subspace containing $\mathbb{R}^2$ (it is the only subspace containing $\mathbb{R}^2$ in fact). Similarly, $V_2 + U = \mathbb{R}^2$. But $V_1 \neq V_2$.

There are many, many other examples.

1C.21 There are many answers. One is

$$W = \{(0, 0, s, t, u) \in \mathbb{F}^5 : s, t, u \in \mathbb{F}\}.$$

Let us prove it. Let $v \in \mathbb{F}^5$. Take $v = (a, b, c, d, e)$. Set

$$\begin{aligned} x &= a \\ y &= b \\ s &= c - a - b \\ t &= d - a + b \\ u &= e - 2a. \end{aligned}$$

Take

$$u = (x, y, x + y, x - y, 2x) = (a, b, a + b, a - b, 2a),$$

and take

$$w = (0, 0, s, t, u) = (0, 0, c - a - b, d - a + b, e - 2a).$$

Certainly $u \in U$ and $w \in W$. A routine calculation shows that $u + w = v$ (omitted).

Next we show that $U \cap W = \{0\}$. For let $v \in U \cap W$. Since $v \in U$, $\exists x, y \in \mathbb{F}$ such that

$$v = (x, y, x + y, x - y, 2x).$$

But $v \in W$, so the first two components must be 0. In other words, $x = y = 0$. It follows that $v = 0$, and so $U \cap W = \{0\}$.

By Prop 1.46, $\mathbb{F}^5 = U \oplus W$.

1C.24 First we show that $\mathbb{R}^{\mathbb{R}} = V_e + V_o$. Let $f \in \mathbb{R}^{\mathbb{R}}$ (in other words, $f : \mathbb{R} \to \mathbb{R}$ is a function). Let

$$\begin{aligned} g(x) &= \frac{1}{2}(f(x) + f(-x)) \text{ and} \\ h(x) &= f - g = \frac{1}{2}(f(x) - f(-x)). \end{aligned}$$

Observe that

$$g(-x) = \frac{1}{2}(f(-x) + f(x)) = \frac{1}{2}(f(x) + f(-x)) = g(x),$$

so g is even. Also,

$$h(-x) = \frac{1}{2}(f(-x) - f(x)) = -\frac{1}{2}(f(x) - f(-x)) = -h(x),$$

so h is odd. Finally,

$$\begin{aligned} g(x) + h(x) &= \frac{1}{2}f(x) + \frac{1}{2}f(-x) + \frac{1}{2}f(x) - \frac{1}{2}f(-x) \\ &= f(x). \end{aligned}$$

Thus $f \in V_e + V_o$. It follows that $\mathbb{R}^{\mathbb{R}} \subseteq V_e + V_o$. The opposite containment is obvious: sums of functions are still functions!

Next we show that $V_e \cap V_o = \{0\}$. Suppose $f \in V_e \cap V_o$. Then

$$\begin{aligned} f(-x) &= f(x) \text{ and} \\ f(-x) &= -f(x). \end{aligned}$$

It follows that $f(x) = -f(x)$, so $2f(x) = 0$, and hence $f(x) = 0$. This holds for all x, and so f is the zero function. Thus $V_e \cap V_o = \{0\}$.

By Prop 1.46, we are done.