Homework 3D Selected Solutions

3D.3 We first prove (a) ⇒ (b). So suppose T is invertible, and let $v_1, \ldots, v_n$ be a basis of V. Suppose $a_1 Tv_1 + \cdots + a_n Tv_n = 0$. Then by linearity,

$$T(a_1 v_1 + \cdots + a_n v_n) = 0.$$

Applying T^{-1} to both sides yields

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

But $v_1, \ldots, v_n$ is a basis, hence linearly independent, so $a_1 = a_2 = \cdots = a_n = 0$. It follows that the Tv_j form a linearly independent list. Since $n = \dim V$ and there are n vectors in the linearly indpendent list $Tv_1, \ldots, Tv_n$, by Prop 2.38 $Tv_1, \ldots, Tv_n$ is a basis of V.

Next, we do (b) ⇒ (c). But this is immediate!

Lastly we do (c) ⇒ (a). Let v_j be as in (c). Define $S : V \to V$ by

$$S(Tv_j) = v_j$$

for all j. By Prop 3.4, there exists a unique linear map S with this property. Observe that $(ST)(v_j) = v_j$; by the uniqueness in Prop 3.4, we have $ST = \mathrm{id}_V$. Also $(TS)(Tv_j) = Tv_j$, so again by the uniqueness in Prop 3.4, $TS = \mathrm{id}_V$. Therefore S is the inverse of T, which is thus invertible.

3D.6 For the backwards direction, suppose $\exists E \in \mathcal{L}(W)$ invertible with $S = ET$. Let $v \in \mathrm{null}(T)$. Then

$$S(v) = (ET)(v) = E(T(v)) = E(0) = 0,$$

where $T(v) = 0$ from $v \in \mathrm{null}(T)$, and $E(0) = 0$ from E being linear. Thus $v \in \mathrm{null}(S)$, and so $\mathrm{null}(T) \subseteq \mathrm{null}(S)$.

Multiplying by the inverse of E, we have $T = E^{-1}S$, so a similar argument shows that $\mathrm{null}(S) \subseteq \mathrm{null}(T)$, and hence the two null spaces are equal.

For the forwards direction, we know that $\mathrm{Im}(S), \mathrm{Im}(T)$ are both subspaces of W, and hence are finite-dimensional. Let $w_1, \ldots, w_m$ be a basis for $\mathrm{Im}(T)$, which we can extend to a basis $w_1, \ldots, w_m, u_1, \ldots, u_n$ for W. By definition of image, $\forall j$, $1 \leq j \leq m$, $\exists v_j \in V$ such that $T(v_j) = w_j$. Let $x_j = S(v_j)$. We would like to define E to satisfy $E(x_j) = w_j$, for then $(ET)(v_j) = S(v_j)$. But to do this, we need to

- make sure the x_j form a linearly independent list;
- extend the x_j to a basis of W; and
- use this basis to define E with Prop 3.4.

To show linear independence, suppose $a_1, \dots, a_m \in \mathbb{F}$ satisfy

$$a_1 x_1 + \cdots + a_m x_m = 0.$$

Then substituting $S(v_j)$ for x_j and using linearity of S, we have

$$S(a_1 v_1 + \cdots + a_m v_m) = 0,$$

and so $a_1 v_1 + \cdots + a_m v_m \in \operatorname{null}(S)$. Since the null spaces are equal, that means $a_1 v_1 + \cdots + a_m v_m \in \operatorname{null}(T)$, and so when we apply T, we get

$$a_1 w_1 + \cdots + a_m w_m = 0.$$

The ws form a basis, hence $a_j = 0$ for all j. Linear independence of $x_1, \dots, x_m$ follows.

Now we extend to a basis $x_1, \dots, x_m, y_1, \dots, y_n$ of W; note that we already know $\dim W = m + n$ from our earlier basis. Define E by

$$\begin{aligned} E(x_j) &= w_j \\ E(y_k) &= u_k \end{aligned}$$

for all j, k, using Prop 3.4. We first show that E is invertible. For if we define $F : W \to W$ by

$$\begin{aligned} F(w_j) &= x_j \\ F(u_k) &= y_k \end{aligned}$$

for all j, k, then one can easily obtain that F is the inverse of E (exercise).

Next we show that $S = ET$. Let $v \in V$. Let $w = T(v)$. Then $\exists b_1, \dots, b_m$ such that $w = b_1 w_1 + \cdots + b_m w_m$, since the w_js form a basis of $\operatorname{Im}(T)$. But then

$$T(b_1 v_1 + \cdots + b_m v_m) = b_1 w_1 + \cdots + b_m w_m = w$$

as well. That means that

$$T(v - (b_1 v_1 + \cdots + b_m v_m)) = T(v) - T(b_1 v_1 + \cdots + b_m v_m) = w - w = 0.$$

Therefore $v - (b_1 v_1 + \cdots + b_m v_m) \in \operatorname{null}(T)$. By hypothesis, we therefore have that

$$v - (b_1 v_1 + \cdots + b_m v_m) \in \operatorname{null}(S).$$

Thus $S(v - (b_1 v_1 + \cdots + b_m v_m)) = 0$, and so

$$\begin{aligned} S(v) &= S(b_1 v_1 + \cdots b_m v_m) \\ &= b_1 x_1 + \cdots + b_m x_m \\ &= E(b_1 w_1 + \cdots + b_m w_m) \\ &= E(T(v)). \end{aligned}$$

Since v was arbitrary, we have $S = ET$.

3D.11 Suppose that S and T are invertible. I claim that $T^{-1}S^{-1}$ is the inverse of ST. Repeatedly using associativity of functional composition, we have

$$\begin{aligned}(T^{-1}S^{-1})(ST) &= T^{-1}(S^{-1}S)T\\ &= T^{-1}\operatorname{id}_V T\\ &= T^{-1}T\\ &= \operatorname{id}_V.\end{aligned}$$

The reverse composition $(ST)(T^{-1}S^{-1}) = \operatorname{id}_V$ is similar.

Now suppose ST is invertible with inverse R. Observe that

$$\begin{aligned}(RS)T &= R(ST)\\ &= \operatorname{id}_V,\end{aligned}$$

and therefore by exercise 3B.19, T is injective. But since $T : V \to V$ and V is finite-dimensional, Prop 3.65 shows that T is invertible. This means that T^{-1} is also invertible. As ST and T^{-1} are invertible, by the first half of this proof, above, their composition $STT^{-1} = S$ is also invertible.

3D.15 Let $w \in V$. Then $\exists a_j \in \mathbb{F}$ such that

$$a_1Tv_1 + \cdots + a_mTv_m = w,$$

and so

$$T(a_1v_1 + \cdots, +a_mv_m) = w.$$

Therefore T is surjective. Since V is finite-dimensional (it has a finite spanning set by hypothesis), by Prop 3.65, T is invertible. Let S be its inverse. Then S is also invertible, and hence surjective. Thus for each $v \in V$, $\exists u \in V$ such that $S(u) = v$. But since the Tv_j span V, we have $\exists b_j \in \mathbb{F}$ such that

$$b_1Tv_1 + \cdots + b_mTv_m = u,$$

so

$$T(b_1v_1 + \cdots + b_mv_m) = u.$$

Applying S to both sides yields

$$b_1v_1 + \cdots + b_mv_m = S(u) = v.$$

Thus $v \in \operatorname{Span}(v_1, \ldots, v_m)$. As v was arbitrary, this shows that v_j span V.