Homework 3 Selected Solutions
Due: Tuesday, February 10

1C.11 Note that there could be uncountably many subspaces, so you cannot index the subspaces with, for instance, integers! Instead, use an unspecified indexing set; I will use Λ (capital lambda). Here's an example: take $V = \mathbb{R}^2$, $\Lambda = \mathbb{R}$, and

$$U_\lambda = \{(x, y) \in V : y = \lambda x\}.$$

In other words, U_λ is the line with slope λ through the origin. There are uncountably many of these, so they cannot be listed.

In any case, given a vector space V, let U_λ be a subspace for $\lambda \in \Lambda$. Let $U = \bigcap_{\lambda \in \Lambda} U_\lambda$. We wish to show that U is a subspace.

First, $0 \in U_\lambda$ for all $\lambda \in \Lambda$, so $0 \in U$.

Second, let $u, v \in U$. Then $u, v \in U_\lambda$ for all $\lambda \in \Lambda$. As U_λ is a subspace, $u + v \in U_\lambda$. Since this holds for all λ, we get $u + v \in U$.

Finally, let $u \in U$ and $a \in \mathbb{F}$. Then $u \in U_\lambda$ for all $\lambda \in \Lambda$. As U_λ is a subspace, $au \in U_\lambda$ as well. Thus $au \in U$.

We now apply Prop. 1.34 to obtain the conclusion.

1C.14 Here's two answers with symbols:

$$\begin{aligned} U + W &= \{(x + y, -x + y, 2x + 2y) \in \mathbb{F}^3 : x, y \in \mathbb{F}\} \\ U + W &= \{(s, t, 2s) \in \mathbb{F}^3 : s, t \in \mathbb{F}\}. \end{aligned}$$

The first one is clearly the case (note that the xs in the two sets are independent, so I changed the second one to y). To see the equality, I used $s = x + y$, $t = -x + y$. Since $x = \frac{1}{2}(s - t)$ and $y = \frac{1}{2}(s + t)$, we can get both containments (exercise).

In words, it is a plane. Specifically, it is the plane perpendicular to the xz-plane, containing the line $z = 2x$.

1C.18 It does! The additive identity is $\{0\}$. For let $U \subseteq V$ be a subspace. By Prop. 1.40, $U \subseteq U + \{0\}$. For the reverse direction, let $v \in U + \{0\}$. Then by definition of $+$, $\exists u \in U$ and $w \in \{0\}$ such that $v = u + w$. But obviously $w = 0$, so in fact $v = u$, and hence $v \in U$. Thus $U + \{0\} = U$.

For additive inverses, only $\{0\}$ has one—itself! For suppose $U \neq \{0\}$ is a subspace. Since U is nontrivial, $\exists u \in U$ with $u \neq 0$. By Prop 1.40, given any other subspace W, $U \subseteq U + W$, so $u \in U + W$. In particular, $U + W \neq \{0\}$. Therefore U cannot have an additive inverse.