Homework 2 Selected Solutions
Due: Tuesday, February 3

1B.3 For existence, let $z = w + (-v)$, and take $x = \frac{1}{3}z$. Then

$$\begin{aligned} v + 3x &= v + z \\ &= v + w - v \\ &= v - v + w \text{ (by commutativity)} \\ &= 0 + w \text{ (by definition of additive inverse)} \\ &= w \text{ (by definition of zero vector).} \end{aligned}$$

For uniqueness, suppose y is another such vector. Then in particular

$$v + 3x = v + 3y.$$

Adding $-v$ to both sides yields $3x = 3y$, and then multiplying by $\frac{1}{3}$ gives $x = y$. Uniqueness follows.

1C.4 Let $U = \{f \in \mathbb{R}^{[0,1]} : \int_0^1 f\,dx = b\}$.

If $b \neq 0$, consider the constant function $g = b$. Certainly $\int_0^1 g\,dx = b$, so $g \in U$. But $\int_0^1 2g\,dx = 2\int_0^1 g\,dx = 2b$. As $2b \neq b$, we get $2g \notin U$. Thus U is not closed under scalar multiplication, and so by Prop. 1.34, U is not a subspace.

Now consider the $b = 0$ case. The zero function is in U. Suppose $f, g \in U$. By definition of U,

$$\int_0^1 f\,dx = 0 = \int_0^1 g\,dx.$$

Then

$$\int_0^1 (f + g)\,dx = \int_0^1 f\,dx + \int_0^1 g\,dx = 0 + 0 = 0,$$

so $f + g \in U$. Thus U is closed under vector addition.

Let $\lambda \in \mathbb{R}$ be a scalar. Then

$$\int_0^1 \lambda f\,dx = \lambda \int_0^1 f\,dx = \lambda \cdot 0 = 0,$$

and thus $\lambda f \in U$. Therefore U is closed under scalar multiplication. By Prop. 1.34, U is a subspace of $\mathbb{R}^{[0,1]}$.

1C.8 There are of course infinitely many examples, but here's a relatively simple one: in $\mathbb{R}^2$, take $U = \{(x, y) : x = 0 \text{ or } y = 0\}$. In other words, U is the union of the coordinate axes.

Let $v \in U$ and $\lambda \in \mathbb{R}$. Since $v \in U$, either $v = (x, 0)$ for some $x \in \mathbb{R}$, or $v = (0, y)$ for some $y \in \mathbb{R}$. In the first case,

$$\lambda v = \lambda(x, 0) = (\lambda x, 0) \in U,$$

and in the second,

$$\lambda v = \lambda(0, y) = (0, \lambda y) \in U.$$

Therefore U is closed under scalar multiplication.

But it is not closed under vector addition! For $(1, 0), (0, 1) \in U$, but

$$(1, 0) + (0, 1) = (1, 1) \notin U.$$

Therefore by Prop. 1.34, U is not a subspace.