Homework 12 Selected Solutions
Due: Tuesday, May 5

3D.24 Let $S : \mathbb{F}^n \to \mathbb{F}^n$ be the linear transformation given by A; that is, if $A = [A_{jk}]$, then

$$S(x) = (\ldots, \sum_{k=1}^{n} A_{jk}x_k, \ldots),$$

where the expression given is for the jth coordinate of $S(x)$. Define T similarly wrtB. We have

$$\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T) = AB = I,$$

where the first equality is by Prop 3.43. Thus $ST = \mathrm{id}_{\mathbb{F}^n}$. By Prop 3.68, since $\dim \mathbb{F}^n = \dim \mathbb{F}^n$ (I know it looks funny), $TS = \mathrm{id}_{\mathbb{F}^n}$. But then

$$I = \mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S) = BA.$$

The conclusion follows.

A. Since

$$\begin{aligned} 1 &= 1 \cdot 1 + 0x + 0x^2 + 0x^3, \\ x + 1 &= 1 \cdot 1 + 1x + 0x^2 + 0x^3, \\ (x+1)^2 &= 1 \cdot 1 + 2x + 1x^2 + 0x^3, \text{ and} \\ (x+1)^3 &= 1 \cdot 1 + 3x + 3x^2 + 1x^3, \end{aligned}$$

we have

$$\mathcal{M}(\mathrm{id}, B_2, B_1) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $\mathcal{M}(\mathrm{id}, B_1, B_2) = \mathcal{M}(\mathrm{id}, B_2, B_1)^{-1}$, which using a computer algebra package is

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have

$$c_{B_1}(2x^3 - x^2 + x + 3) = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 2 \end{bmatrix},$$

so

$$\begin{aligned} c_{B_2}(2x^3 - x^2 + x + 3) &= \mathcal{M}(\mathrm{id}, B_1, B_2)c_{B_1}(2x - x^2 + x + 3) \\ &= \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 9 \\ -7 \\ 2 \end{bmatrix}. \end{aligned}$$

Therefore

$$2x^3 - x^2 + x + 3 = -1 + 9(x+1) - 7(x+1)^2 + 2(x+1)^3.$$

B. We have

$$\begin{aligned} D(\cos^2 x) &= 0\cos^2 x + 0\sin^2 x - 2\cos x \sin x + 0\cos x + 0\sin x, \\ D(\sin^2 x) &= 0\cos^2 x + 0\sin^2 x + 2\cos x \sin x + 0\cos x + 0\sin x, \\ D(\cos x \sin x) &= 1\cos^2 x - 1\sin^2 x + 0\cos x \sin x + 0\cos x + 0\sin x \\ D(\cos x) &= 0\cos^2 x + 0\sin^2 x + 0\cos x \sin x + 0\cos x - 1\sin x, \text{ and} \\ D(\sin x) &= 0\cos^2 x + 0\sin^2 x + 0\cos x \sin x + 1\cos x + 0\sin x. \end{aligned}$$

Therefore our matrix is

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$