

## Homework 11 Selected Solutions

Due: Tuesday, April 28

3D.12 We have  $S(TU) = I$ , so  $S$  is invertible by Prop 3.68. By the same reasoning,  $(ST)U = I$  implies that  $U$  is invertible. Multiplying on the left by  $S^{-1}$  and on the right by  $U^{-1}$  yields

$$T = S^{-1}U^{-1}.$$

Then

$$\begin{aligned} (US)T &= (US)S^{-1}U^{-1} \\ &= U(SS^{-1})U^{-1} \\ &= UIU^{-1} \\ &= I, \end{aligned}$$

and so by Prop 3.68,  $T$  is invertible with inverse  $US$ . This is equivalent to the conclusion given.

3D.19 First suppose  $T = \lambda \text{id}_V$  for some  $\lambda \in \mathbb{F}$ . Let  $B = (v_1, \dots, v_n)$  be a basis for  $V$ . Then  $T(v_j) = \lambda v_j$  for all  $j$ , and therefore the  $j$ th column of  $\mathcal{M}(T, B)$  is  $\lambda e_j$ . This implies that  $\mathcal{M}(T, B)$  is  $\lambda I_n$ . Since the matrix does not depend on  $B$ , the claim follows.

Now suppose the matrix of  $T$  does not depend on the choice of basis. Let  $B$  be as above, and let  $\mathcal{M}(T, B) = [A_{jk}]$ ,  $1 \leq j, k \leq n$ . Let

$$B_i = (v_1, \dots, v_{i-1}, 2v_i, v_{i+1}, \dots, v_n).$$

Write  $w_j$  for the  $j$ th element of  $B_i$ ; so  $w_j = v_j$  unless  $j = i$ , in which case  $w_j = 2v_j$ . From the definition of  $A_{jk}$ , we have

$$T(v_j) = \sum_{t=1}^n A_{tj} v_t.$$

Therefore if  $j \neq i$ ,

$$T(w_j) = \sum_{t \neq i} A_{tj} w_t + \frac{1}{2} A_{ij} w_i.$$

This means that the  $j$ th column of  $\mathcal{M}(T, B_i)$  is the same as the  $j$ th column of  $\mathcal{M}(T, B)$ , except for the  $i$ th row. The entry in the  $i$ th row is half the value of the entry in  $\mathcal{M}(T, B_i)$ . But since the matrices are the same by hypothesis, we get  $A_{ij} = \frac{1}{2} A_{ij}$ . This can only occur if  $A_{ij} = 0$ . The only condition we had on  $i, j$  was  $i \neq j$ . Therefore  $\mathcal{M}(T, B)$  is a diagonal matrix.

It remains to show that the diagonal entries are all the same. Let  $B' = (v_2, v_1, v_3, v_4, \dots, v_n)$ . Define  $u_j$  to be the  $j$ th entry of  $B'$ . Thus  $u_1 = v_2$  and  $u_2 = v_1$ . Since

$$T(v_1) = A_{11}v_1,$$

we have  $T(u_2) = A_{11}u_2$ . Therefore the  $(2, 2)$  entry of  $\mathcal{M}(T, B')$  is  $A_{11}$ . Since  $\mathcal{M}(T, B) = \mathcal{M}(T, B')$ , we conclude that  $A_{11} = A_{22}$ . By a similar argument (namely, using the basis that swaps  $v_1$  and  $v_j$ ) we see that  $A_{11} = A_{jj}$  for all  $j$ . Therefore  $\mathcal{M}(T, B)$  is a diagonal matrix with all entries the same. Thus  $\mathcal{M}(T, B) = \lambda I_n$  for some  $\lambda \in \mathbb{F}$ , and so  $T(v_j) = \lambda v_j$  for all  $j$ . In particular, by the uniqueness part of the Linear Map Lemma,  $T$  must equal  $\lambda \text{id}_V$ .

3D.21 Let  $A = [A_{jk}]$ ,  $x = (x_1, \dots, x_n)$  written as a column vector, and define

$$\begin{aligned} T : \mathbb{F}^n &\rightarrow \mathbb{F}^n \\ x &\mapsto Ax. \end{aligned}$$

We know that  $T$  is a linear transformation from 3A (and the proposition we proved in class). Now, note that the condition in (a) is identical to saying, "if  $T(x) = 0$ , then  $x = 0$ ". This condition is equivalent to  $\text{null}(T) = \{0\}$ , which we know is equivalent to  $T$  being injective. Meanwhile, the condition in (b) is identical to saying, " $\forall c \in \mathbb{F}^n$ ,  $\exists x \in \mathbb{F}^n$  such that  $T(x) = c$ ," which is the definition of  $T$  being surjective. Finally, since  $V$  is finite-dimensional, by Prop 3.65 we know that  $T$  is injective if and only if it is surjective. Therefore (a) is true if and only if (b) is true.