Homework 10 Selected Solutions
Due: Thursday, April 16

10 There are infinitely many answers; in fact, if you pick two random matrices, the odds are overwhelmingly high that they will give a correct example. But I will pick simple ones. Take

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A routine calculation shows that

$$AB = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \text{ while } BA = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

12 Let's say $A = [A_{jk}]$ is $m \times n$, B is $n \times p$, and C is $p \times q$. We first define linear transformations

$$\begin{aligned} R :& \mathbb{F}^n \to \mathbb{F}^m \\ S :& \mathbb{F}^p \to \mathbb{F}^n \\ T :& \mathbb{F}^q \to \mathbb{F}^p \end{aligned}$$

so that, with respect to the standard bases, $\mathcal{M}(R) = A$, $\mathcal{M}(S) = B$, and $\mathcal{M}(T) = C$. For R, define

$$R(x_1, \ldots, x_m) = \left(\ldots, \sum_{k=1}^{m} A_{jk} x_k, \ldots \right);$$

that is, the jth entry is $\sum_k A_{jk} x_k$. By a proposition we covered in class, R is a linear map. (This is the construction on page 53 of the text, under the heading "from $\mathbb{F}^n$ to $\mathbb{F}^m$." For whatever reason, the text never proved that this is linear, but we did it in class.) Additionally, by construction $\mathcal{M}(R) = A$. We construct S and T in a similar manner.

Now functional composition is associative, so

$$(RS)T = R(ST).$$

Thus $\mathcal{M}((RS)T) = \mathcal{M}(R(ST))$. By Prop 3.43,

$$\mathcal{M}((RS)T) = \mathcal{M}(RS)\mathcal{M}(T) = (\mathcal{M}(R)\mathcal{M}(S))\mathcal{M}(T) = (AB)C.$$

Similarly,

$$\mathcal{M}(R(ST)) = \mathcal{M}(R)\mathcal{M}(ST) = \mathcal{M}(R)(\mathcal{M}(S)\mathcal{M}(T)) = A(BC).$$

Therefore $(AB)C = A(BC)$.

Note that problem 10 can be done in a similar manner.

15 First notice that both sides are the same size: $p \times n$. We compute the jk entry on each side. We have $((AC)^T)_{j,k} = (AC)_{k,j}$ by definition of transpose. By definition of matrix multiplication, this is

$$\sum_{t=1}^{n} A_{k,t} C_{t,j}.$$

On the other hand, $(C^T A^T)_{j,k} = \sum_{t=1}^{n} (C^T)_{j,t} (A^T)_{t,k}$. By definition of transpose, $(C^T)_{j,t} = C_{t,j}$ and $(A^T)_{t,k} = A_{k,t}$. Thus

$$(C^T A^T)_{j,k} = \sum_{t=1}^{n} C_{t,j} A_{k,t} = \sum_{t=1}^{n} A_{k,t} C_{t,j} = ((AC)^T)_{j,k}.$$

Note that the entries in the summations are scalars, so multiplication is commutative.