Homework 1 Selected Solutions
Due: Tuesday, January 27

1A.6 Let $\alpha = a + bi$ with $a, b \in \mathbb{R}$ not both zero. As stated in class, we can let

$$\beta = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i = \frac{a-bi}{a^2+b^2}.$$

Notice that the denominator is nonzero by our hypothesis on a and b. Next, we multiply through to get

$$\begin{aligned}\alpha\beta &= (a+bi) \cdot \frac{a-bi}{a^2+b^2} \\ &= \frac{(a+bi)(a-bi)}{a^2+b^2} \\ &= \frac{a^2-b^2i^2}{a^2+b^2} \\ &= \frac{a^2+b^2}{a^2+b^2} = 1.\end{aligned}$$

The existence claim therefore holds. For uniqueness, if β, β' are both inverses for α, then

$$\beta = 1 \cdot \beta = (\beta'\alpha)\beta = \beta'(\alpha\beta) = \beta' \cdot 1 = \beta'.$$

Thus $\beta = \beta'$, and uniqueness follows.

1B.2 If $a = 0$, then we are done. So suppose $a \neq 0$. Then since all nonzero elements of $\mathbb{F}$ have multiplicative inverses, $\exists b \in \mathbb{F}$ such that $ab = 1$. Thus

$$b(av) = (ba)v = 1v = v.$$

Here, we are using associativity of scalar multiplication and that 1 is the multiplicative identity.

On the other hand, $b \cdot 0 = 0$ by Prop. 1.31. Thus when we multiply both sides of $av = 0$ by b, we get $v = 0$, proving the claim.

1B.5 I'll define Definition 1.20′ to be Definition 1.20, but with the *additive inverse* statement replaced with "$\forall v \in V$, $0 \cdot v = 0$." I'll call V a *wector space* if it satisfies Definition 1.20′. We want to show that V is a vector space if and only if it is a wector space.

Suppose V is a vector space. Then certainly it satisfies all of the conditions to be a wector space aside from the new one: "$\forall v \in V$, $0 \cdot v = 0$." But by Prop. 1.30, V also satisfies this last condition. Therefore V is a wector space.

Suppose V is a wector space. Again, it immediately satisfies all of the conditions to be a vector space except possible the additive inverse property. For that, let $v \in V$. I claim that $(-1) \cdot v$ is an additive inverse; for

$$\begin{aligned} v + (-1) \cdot v &= 1 \cdot v + (-1) \cdot v \text{ (by the multiplicative identity property)} \\ &= (1 + (-1)) \cdot v \text{ (by distributivity)} \\ &= 0 \cdot v \\ &= 0 \end{aligned}$$

where the last equality follows from the definition of wector space. Since

$$v + (-1) \cdot v = 0,$$

by definition of additive inverse, $(-1) \cdot v$ is indeed an additive inverse for v. Therefore V is a vector space.