Name:

Math 364: Exam 2

April 21, 2026

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. You may use any result from the text, but not homework problems; ask me if you are unsure if a result was proven in the text. Avoid using the back of the page; instead, there is an extra sheet at the end that you can use. Calculators are not allowed.

1. Short answer questions. Unless instructed, you do not have to show work or justify your answers. Remember to specify a function, you must give the domain, the codomain, and a method of evaluating the function (like a formula).

   (a) (10 points) Give an example of a linear map T with $\dim(\text{null}(T)) = 1$.

   Solution: Of course there are infinitely many. I give three.

   $$\begin{aligned} T_1 : \mathbb{R}^2 &\to \mathbb{R} \\ T_1(x, y) &= x. \end{aligned}$$

   The null space is $\text{Span}(e_2)$. A simpler one:

   $$\begin{aligned} T_2 : \mathbb{R} &\to \{0\} \\ x &\mapsto 0. \end{aligned}$$

   A last example is the map from problem 2.

   (b) (10 points) Give an example of a linear map T for which $\text{null}(T)$ is infinite-dimensional, but $T \neq 0$.

   Solution: Say $T : V \to W$. Since $\text{null}(T) \subseteq V$, V must be infinite-dimensional. One example is

   $$\begin{aligned} T : \mathbb{R}^\infty &\to \mathbb{R} \\ (x_1, x_2, \ldots) &\mapsto x_1. \end{aligned}$$

   Another example:

   $$\begin{aligned} T : \mathscr{P}(\mathbb{R}) &\to \mathbb{R} \\ p(x) &\mapsto p(0). \end{aligned}$$

   In fact, any example where V is infinite-dimensional, W is finite-dimensional, and $T \neq 0$ will work.

   (c) (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection across the line $y = -x$. Compute the matrix $\mathcal{M}(T)$ with respect to the standard basis for $\mathbb{R}^2$.

   Solution: We have $T(e_1) = -e_2 = 0e_1 - 1e_2$ and $T(e_2) = -e_1 = -1e_1 + 0e_2$, so the matrix is

   $$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

(d) (10 points) Let (v_1, v_2, v_3) be a basis for the vector space V, and (w_1, w_2) a basis for the vector space W. Suppose that with respect to these two bases, the matrix of a linear map $T : V \to W$ is

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Now we change our bases: pick (v_1, v_3, v_2) as our basis for V, and (w_2, w_1) as our basis for W. What is the new matrix for T?

Solution: From the given matrix, we have

$$\begin{aligned} T(v_1) &= 1w_1 + 4w_2 \\ T(v_2) &= 2w_1 + 5w_2 \\ T(v_3) &= 3w_1 + 6w_2. \end{aligned}$$

Therefore

$$\begin{aligned} T(v_1) &= 4w_2 + 1w_1 \\ T(v_3) &= 6w_2 + 3w_1 \\ T(v_2) &= 5w_2 + 2w_1, \end{aligned}$$

and so the new matrix is

$$\begin{bmatrix} 4 & 6 & 5 \\ 1 & 3 & 2 \end{bmatrix}.$$

2. Let $T : \mathscr{P}_2(\mathbb{R}) \to \mathbb{R}^2$ be given by

$$T(p(x)) = (p(1), p'(1)).$$

(a) (15 points) Show that T is linear.

Solution: Let $p, q \in \mathscr{P}_2(\mathbb{R})$. Then

$$\begin{aligned} T(p+q) &= ((p+q)(1), (p+q)'(1)) \\ &= (p(1) + q(1), (p' + q')(1)) \text{ by defn of fn addition and additivity of deriv.} \\ &= (p(1) + q(1), p'(1) + q'(1)) \\ &= (p(1), p'(1)) + (q(1), q'(1)) \\ &= T(p) + T(q). \end{aligned}$$

Let $c \in \mathbb{R}$. Then

$$\begin{aligned} T(cp) &= ((cp)(1), (cp)'(1)) \\ &= (cp(1), cp'(1)) \\ &= c(p(1), p'(1)) \\ &= cT(p). \end{aligned}$$

Therefore T is linear.

(b) (10 points) Let $B = (1, x, x^2)$ and $B' = (e_1, e_2)$ (in $\mathbb{R}^2$). Compute $\mathcal{M}(T, B, B')$.

Solution: We have

$$T(1) = (0,0) = 0e_1 + 0e_1,$$
$$T(x) = (1,0) = 1e_1 + 0e_2,$$
$$T(x^2) = (1,2) = 1e_1 + 2e_2.$$

Therefore the matrix is

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

3. (15 points) Suppose V and W are vector spaces, $T : V \to W$ an invertible linear map, and $v_1, v_2, \dots, v_n \in V$ a linearly independent list. Show that $T(v_1), \dots, T(v_n)$ is linearly independent.

Solution: Suppose $a_1, \dots, a_n$ are scalars satisfying

$$a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n) = 0.$$

Since T is invertible, it has an inverse T^{-1}. Applying T^{-1} to both sides of the above equation, we get

$$T^{-1}(a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n)) = T^{-1}(0).$$

Using the linearity of T^{-1} and that linear maps send 0 to 0, we get

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

But $v_1, \dots, v_n$ is linearly independent, so $a_1 = a_2 = \cdots = a_n = 0$. Therefore our original list $T(v_1), \dots, T(v_n)$ is linearly independent.

4. Let V and W be finite-dimensional vector spaces with $\dim V = \dim W$. Suppose $S \in \mathcal{L}(V, W)$, $T \in \mathcal{L}(W, V)$, and $ST = \mathrm{id}_W$.

   (a) (15 points) Show that T is an isomorphism.

Solution: Suppose $w \in \mathrm{null}(T)$. Then $T(w) = 0$. Applying S to both sides, we get $S(T(w)) = S(0)$, so $\mathrm{id}_W(w) = 0$, and therefore $w = 0$. Therefore $\mathrm{null}(T) = \{0\}$, and hence T is injective.

By Rank-Nullity,

$$\dim W = \dim \mathrm{null}(T) + \dim \mathrm{Im}(T).$$

We have $\dim(\mathrm{null}(T)) = 0$ and $\dim W = \dim V$, so

$$\dim V = \dim \mathrm{Im}(T).$$

Since $\mathrm{Im}(T)$ is a subspace of V of the same dimension, we must have $\mathrm{Im}(T) = V$, so T is surjective.

Since T is a bijective linear map, it is an isomorphism.

   (b) (10 points) Show by example that if we remove the hypothesis that $\dim V = \dim W$, then we may have $ST = \mathrm{id}_W$ and T *not* an isomorphism.

Solution: Here's the example from class: take $V = W = \mathbb{R}^\infty$, T the right shift operator, and S the left shift operator. We have for $w = (w_1, w_2, \ldots) \in W$,

$$\begin{aligned} (ST)(w) &= (S(T(w)) \\ &= S(0, w_1, w_2, \ldots) \\ &= (w_1, w_2, \ldots) \\ &= w, \end{aligned}$$

and hence $ST = \mathrm{id}_W$. But T is not surjective since (for instance) $(1, 0, 0, \ldots)$ is not in $\mathrm{Im}(T)$.

Here's another example: $V = W = \mathscr{P}(\mathbb{R})$, $T(p) = \int_0^x p(t)\,dt$, $S(p) = p'$. One of the Fundamental Theorems of Calculus shows that $ST = \mathrm{id}_W$. But the constant 1 is not in $\mathrm{Im}(T)$. (In fact, $q \in \mathrm{Im}(T)$ if and only if $q(0) = 0$.)