Name:

Math 364: Exam 1

March 10, 2026

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Avoid using the back of the page; instead, there is an extra sheet at the end that you can use. Calculators are not allowed.

How often did you use AI on your homework? (0 = never, 4 = on every problem)

0 1 2 3 4

On average, how much help did you use from AI? (0 = none, 1 = just a hint, 4 = entire solution)

0 1 2 3 4

1. Short answer questions. Unless instructed, you do not have to show work or justify your answers.

(a) (5 points) Let $v_1, \ldots, v_m$ be a list of vectors. Give the rigorous definition of the list being *linearly dependent*.

Solution: $\exists a_1, \ldots, a_m \in \mathbb{F}$, not all zero, such that $a_1 v_1 + \cdots + a_m v_m = 0$.

(b) (5 points) True or false: $\operatorname{Span}((1,1),(1,0)) = \mathbb{R}^2$.

Solution: True. We have $(0,1) = (1,1) - (1,0)$ is in the span. But then the span contains $(1,0),(0,1)$, which forms a basis of $\mathbb{R}^2$. So the span is everything.

(b) ______________

(c) (5 points) True or false: $\mathbb{R}^2 = \operatorname{Span}((1,1)) \oplus \operatorname{Span}((1,-1))$.

Solution: True. We have

$$(x,y) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right) + \left(\frac{x-y}{2}, -\frac{x-y}{2}\right),$$

so $\mathbb{R}^2$ is the sum of the spans. Any vector in both spans must look like (t,t) and $(s,-s)$ simultaneously. Comparing the first coordinate, we get $s = t$. Comparing the second, we get $s = -t$. Thus $t = -t$, so $t = 0$, and hence our vector is $(0,0)$. Thus the intersection is trivial. It follows that the sum is a direct sum.

(c) ______________

(d) (5 points) True or false: in $\mathbb{R}^2$, $((2,3),(4,6))$ is linearly independent.

Solution: False: $2(2,3) - (4,6) = 0$.

(d) ______________

(e) (5 points) True or false: in $\mathbb{R}^2$, $((1,-1),(0,2))$ is linearly independent.

Solution: True; they are not parallel.

(e) ______________

2. (20 points) Show that

$$U = \{(x, y, z) : 2x - y + 3z = 0\}$$

is a subspace of $\mathbb{R}^3$.

Solution: Note that $2 \cdot 0 - 0 + 3 \cdot 0 = 0$, and so $0 = (0, 0, 0) \in U$.

Next, suppose $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ are in U. By definition of U,

$$2x_1 - y_1 + 3z_1 = 0 = 2x_2 - y_2 + 3z_2.$$

Adding the far left and right sides yields

$$0 = 2x_1 - y_1 + 3z_1 + 2x_2 - y_2 + 3z_2 = 2(x_1 + x_2) - (y_1 + y_2) + 3(z_1 + z_2).$$

It follows that $u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ is in U by definition of U.

Let u be as above, and $a \in \mathbb{R}$. Then $au = (ax_1, ay_1, az_1)$, and

$$2(ax_1) - (ay_1) + 3(az_1) = a(2x_1 - y_1 + 3z_1) = a \cdot 0 = 0.$$

Therefore $au \in U$.

Since U contains 0, is closed under addition, and is closed under scalar multiplication, it is a subspace.

3. (20 points) Give an example of subspaces U, W of a vector space V for which $U \cup W$ is not a subspace. Prove your answer.

> **Solution:** See the HW 3 solutions.

4. (20 points) In $\mathbb{R}^4$, let

$$U = \{(x_1, x_2, x_3, x_4) : x_1 - x_2 = 0 \text{ and } x_1 + x_2 - x_3 = 0\}.$$

You may assume that U is a subspace. Find a basis for U, and prove your answer.

> **Solution:** I claim that $B = ((1,1,2,0), (0,0,0,1))$ is a basis. The two vectors are clearly not parallel, hence B is linearly independent.
>
> By plugging in the two vectors into both equations, we see that they are satisfied. Since the two vectors lie in U, we see that $\text{Span}(B) \subseteq U$.
>
> Now let $v \in U$, so $v = (x_1, x_2, x_3, x_4)$ for some $x_1, x_2, x_3, x_4 \in \mathbb{R}$. Since v satisfies the first equation, we have $x_2 = x_1$. The second equation therefore becomes $2x_1 - x_3 = 0$, or $x_3 = 2x_1$. Thus
>
> $$v = (x_1, x_1, 2x_1, x_4) = x_1(1,1,2,0) + x_4(0,0,0,1).$$
>
> Thus $v \in \text{Span}(B)$. Therefore $U \subseteq \text{Span}(B)$.
>
> Putting this all together, we have $U = \text{Span}(B)$, and since B is linearly independent, it is a basis for U.

5. (20 points) In $\mathbb{R}^3$, let $U = \{(x,y,z) : x + y + z = 0\}$ and $W = \text{Span}((1,1,1))$. Prove that

$$\mathbb{R}^3 = U \oplus W.$$

> **Solution:** First we show that $U \cap W = \{0\}$. Any vector in W is of the form $c(1,1,1) = (c,c,c)$ for some scalar c. To also lie in U, we must have
>
> $$c + c + c = 3c = 0,$$
>
> so $c = 0$. Thus $c(1,1,1) = 0$, and so $U \cap W = \{0\}$.
>
> Next we have to show that $\mathbb{R}^3 = U + W$. Here's the slick solution: Given $v = (x,y,z) \in \mathbb{R}^3$, let $c = \frac{1}{3}(x+y+z)$, $w = c(1,1,1)$, and $u = v - w$. Certainly $v = u + w$ and $w \in W$. As for u, we have
>
> $$u = (x-c, y-c, z-c)$$
>
> and so
>
> $$\begin{aligned}(x-c) + (y-c) + (z-c) &= (x+y+z) - 3c \\ &= 0.\end{aligned}$$
>
> Therefore $u \in U$. The claim follows.
>
> Now for the more straighforward solution. Observe that $(1,-1,0), (1,0,-1) \in U$. (They form a basis, but we don't actually need this fact.) For $v = (x,y,z) \in \mathbb{R}^3$, let
>
> $$\begin{aligned}a &= \frac{1}{3}x - \frac{2}{3}y + \frac{1}{3}z \\ b &= \frac{1}{3}x + \frac{1}{3}y - \frac{2}{3}z, \text{ and} \\ c &= \frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z.\end{aligned}$$

Certainly $a(1,-1,0)+b(1,0,-1) \in U$ and $c(1,1,1) \in W$. A routine calculation also shows that

$$a(1,-1,0)+b(1,0,-1)+c(1,1,1)=(x,y,z)=v.$$

6. (20 points) In $\mathscr{P}_3(\mathbb{R})$, let

$$U=\{p \in \mathscr{P}_3(\mathbb{R}) : p(1)=0\}.$$

You may assume that U is a subspace. Compute $\dim U$, and prove your answer.

Solution: The dimension is 3. We need to find a basis. I claim that a basis is given by

$$(x-1), x(x-1), x^2(x-1).$$

Certainly these three are in U. To show linear independence, observe that if

$$a(x-1)+bx(x-1)+cx^2(x-1)=0,$$

then the coefficient of x^3 on the left is c. Therefore $c=0$. Now we consider the coefficient of x^2 on the left, which is b. Therefore $b=0$. Finally, that gives $a(x-1)=0$, so $a=0$. Linear independence follows.

We have $\dim U \geq 3$. Clearly $p(x)=x \notin U$, since $p(1)=1 \neq 0$. Therefore $U \neq \mathscr{P}_3(\mathbb{R})$. It follows that $\dim U \neq 4$, and hence $\dim U=3$.