MATH 350 Assignment 6 Solutions

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4.16

Proof. Let a and b be two integers of the same parity.
We wish to show $a + b$ is even. Let us approach this with cases.
Case 1: a and b are both even.
Thus by definition of even $a = 2j$ and $b = 2k$ for some $j, k \in \mathbb{Z}$.
Observe

$$\begin{aligned} a + b &= 2j + 2k \\ &= 2(j + k). \end{aligned}$$

Since $(j + k) = h$ is an integer, $(a + b) = 2h$ is even by definition.
Case 2: a and b are both odd.
Thus by definition of odd $a = 2m + 1$ and $b = 2\ell + 1$ for some $m, \ell \in \mathbb{Z}$.
Observe

$$\begin{aligned} a + b &= (2m + 1) + (2\ell + 1) \\ &= 2m + 2\ell + 2 \\ &= 2(m + \ell + 1). \end{aligned}$$

Notice that $(m + \ell + 1) = p$ is an integer, thus $(a + b) = 2p$.
Therefore, $(a + b)$ is even by definition. □

4.26

Proof. We want to show that every odd integer is the difference between two squares.
Let x be an arbitrary odd integer, so $x = 2k + 1$ where $k \in \mathbb{Z}$.
Consider $a = k^2$ and $b = (k + 1)^2$. Note that these are both perfect squares.
Observe

$$\begin{aligned} (k + 1)^2 - k^2 &= k^2 + 2k + 1 - k^2 \\ &= 2k + 1 \\ &= x. \end{aligned}$$

Thus every odd integer can be expressed as a difference of perfect squares. □

5.11

Proof. We will prove this by showing the contrapositive.
Suppose x is odd *and* y is even. We will show that $x^2(y+3)$ is odd.
Thus by definition of odd and even respectively, $x = 2k+1$ and $y = 2j$ for some $j, k \in \mathbb{Z}$. Observe

$$\begin{aligned} x^2(y+3) &= (2k+1)^2\big((2j)+3\big) \\ &= (4k^2+4k+1)(2j+3) \\ &= (8jk^2+12k^2+8jk+12k+2j+3) \\ &= (8jk^2+12k^2+8jk+12k+2j+2+1) \\ &= 2(4jk^2+6k^2+4jk+6k+j+1)+1 \end{aligned}$$

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If we let $\ell = 4jk^2+6k^2+4jk+6k+j+1$, and note that $\ell \in \mathbb{Z}$, we see that $x^2(y+3) = 2\ell+1$. Thus $x^2(y+3)$ is odd by definition.
Thus the contapositive of the statementis true and we conclude that our original statement holds. □

5.15

Proof. Suppose $x \in \mathbb{Z}$, and that x^3-1 is even. We want to show that x is odd.
We will prove this by instead proving the contrapositive.
Thus let us assume that x is even and we will demonstrate the x^3-1 is odd.
By definition of even, $x = 2k$ for $k \in \mathbb{Z}$. Observe

$$\begin{aligned} x^3-1 &= (2k)^3-1 \\ &= 8k^3-1 \\ &= 2(4k^3)-1 \\ &= 2(4k^3)-2+1 \\ &= 2(4k^3-1)+1. \end{aligned}$$

Notice that $4k^3-1$ is an integer. Let $j = 4k^3-1$.
Thus, $x^3-1 = 2j+1$ is odd by definition.
Therefore, the contrapositive is true and we can conclude that the original statement is also true. □

5.19

Proof. Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$. Suppose $a \equiv b \mod n$ and $a \equiv c \mod n$.
We wish to show that $c \equiv b \mod n$.
By definition of congruence modulo n,

$$n \mid (a-b) \text{ and } n \mid (a-c).$$

Then we can apply the definition of divides to see that for some $j, k \in \mathbb{Z}$

$$nj = (a-b) \text{ and } nk = (a-c).$$

Subtraction yields

$$nj + b = a \text{ and } nk + c = a.$$

Thus $nj + b = nk + c$. Observe

$$nj - nk = c - b.$$

Therefore, $n(j-k) = (c-b)$, where $(j-k) = \ell$ is an integer.
From this we see that $n\ell = (c-b)$, and by definition of divides we get

$$n \mid (c-b).$$

Applying the definition of congruence modulo n allows us to conclude that

$$c \equiv b \mod n$$

as desired. □