Name:

Math 350: Exam 5

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Calculators are not allowed.

You may use any result from the chapters covered in the text or from lecture. You may not use the results of homework or worksheet problems.

1. (5 pts) How many subsets are there from the set $\{5, 10, 15, 20, 25\}$ which have exactly 2 elements? Give a single number for you answer.

Solution: There are $\binom{5}{2} = \frac{5 \cdot 4}{2} = 10$.

2. (5 pts) Find the coefficient of x in $(x+2)^4$. Give your answer as a single number.

Solution: This coefficient is $\binom{4}{1} \cdot 2^3 = 4 \cdot 8 = 32$.

3. (20 pts) Let A, B, C be sets. Prove that

$$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C).$$

Solution: Let $x \in A \times (B \cap C)$. By definition of Cartesian product, $x = (u, v)$ for some $u \in A$ and $v \in B \cap C$. By definition of intersection, $v \in B$ and $v \in C$. As $u \in A$ and $v \in B$, $(u, v) \in A \times B$. As $u \in A$ and $v \in C$, $(u, v) \in A \times C$. By definition of intersection, $x = (u, v) \in (A \times B) \cap (A \times C)$. The claim follows.

4. (20 pts) Use induction to show that $\forall n \geq 1$,

$$1 + 2 + \cdots + 2^n = 2^{n+1} - 1.$$

Solution: The base case is $n = 1$. On the left side, we get $1 + 2^1 = 3$. On the right, we get $2^{1+1} - 1 = 4 - 1 = 3$. These are equal, so the base case holds.

Now for the inductive step. Suppose we know that

$$1 + 2 + \cdots + 2^n = 2^{n+1} - 1$$

for some $n \geq 1$. Adding 2^{n+1} to both sides, we get

$$\begin{aligned} 1 + 2 + \cdots + 2^n + 2^{n+1} &= 2^{n+1} + 2^{n+1} - 1 \\ &= 2 \cdot 2^{n+1} - 1 \\ &= 2^{n+2} - 1 \\ &= 2^{(n+1)+1} - 1. \end{aligned}$$

This proves the inductive step. By induction, the claim holds.