Name:

Math 350: Exam 4

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Calculators are not allowed.

You may use any result from the chapters covered in the text or from lecture. You may not use the results of homework or worksheet problems.

1. (10 pts) Let $f(x) = x^2 + 101x + 1532$. Compute the remainder of $f(10003)$ on division by 5.

Solution: We have $10003 \equiv 3 \pmod 5$, so $10003^2 \equiv 3^2 \equiv 9 \equiv 4 \pmod 5$. Similarly, $101 \cdot 10003 \equiv 1 \cdot 3 = 3 \pmod 5$, and $1532 \equiv 2 \pmod 5$. Thus

$$\begin{aligned} f(10003) &\equiv 4 + 3 + 2 \pmod 5 \\ &\equiv 9 \pmod 5 \\ &\equiv 4 \pmod 5. \end{aligned}$$

Since every number is congruent to its remainder, and no other remainder, mod 5, the remainder of $f(10003)$ on division by 5 must be 4.

2. (20 pts) Let $x \in \mathbb{R}$. Prove that if x^2 is irrational, then x is irrational.

Solution: We can use either contradiction or contraposition; I prefer contraposition. So we will prove that if x is rational, then x^2 is rational. Suppose x is rational. By definition of rational, $\exists a, b \in \mathbb{Z}$, $b \neq 0$, such that $x = \frac{a}{b}$. Hence $x^2 = \frac{a^2}{b^2}$. As $a^2, b^2 \in \mathbb{Z}$, we have that x^2 is rational by definition of rational. The contrapositive is true, and hence the original statement is also true.

3. (20 pts) Let $a, b, c \in \mathbb{N}$. Prove that if $a \nmid bc$, then $a \nmid b$.

Solution: We use contradiction or contraposition. I prefer contraposition. So we instead prove that if $a \mid b$, then $a \mid bc$. Suppose $a \mid b$. By definition of divides, $\exists k \in \mathbb{Z}$ such that $b = ak$. Multiplying through by c yields $bc = akc$. Let $\ell = kc$, so that $bc = a\ell$. Note that $\ell \in \mathbb{Z}$. By definition of divides, $a \mid bc$. This proves the contrapositive, and hence the original statement is also true.