Name:

Math 350: Exam 3

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Calculators are not allowed.

You may use any result from the chapters covered in the text or from lecture. You may not use the results of homework or worksheet problems.

1. Consider 6-digit strings of 0s and 1s; for instance, 101010.

   (a) (5 pts) How many are there?

   Solution: For each digit, there are 2 choices, for a total of $2^6 = 64$ strings.

   (b) (5 pts) How many are there with last digit 0?

   Solution: The last of the 6 digits is fixed, leaving 5 digits to choose. This gives $2^5 = 32$ strings.

   (c) (5 pts) How many are there for which *either* the 2nd or 4th digit is 1?

   Solution: We count the opposite: strings where both the 2nd and 4th digit are 0. We choose the remaining 4 digits to yields $2^4 = 16$ strings. We subtract from the total number of strings to yield $64 - 16 = 48$ strings.

2. (5 pts) Compute $\dfrac{10!}{7!}$.

   Solution: This is

$$\begin{aligned}\frac{10!}{7!} &= \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} \\ &= 10 \cdot 9 \cdot 8 \\ &= 720.\end{aligned}$$

3. (15 pts) Prove that if n is even, then $n^2 + n + 1$ is odd.

   Solution: Suppose n is even. Then $\exists k \in \mathbb{Z}$ such that $n = 2k$ by definition of even. Then

$$\begin{aligned}n^2 + n + 1 &= (2k)^2 + (2k) + 1 \\ &= 4k^2 + 2k + 1 \\ &= 2(2k^2 + k) + 1.\end{aligned}$$

   Let $\ell = 2k^2 + k$; observe that $\ell \in \mathbb{Z}$. Thus $n^2 + n + 1 = 2\ell + 1$ is odd by definition of odd, proving the cliam.

4. (15 pts) Prove that if $n, a, b \in \mathbb{N}$ satisfy

$$n \mid (a - b), \text{ then } n \mid (a^2 - b^2).$$

   Solution: Suppose $n \mid (a - b)$. By definition of divides, $\exists k \in \mathbb{Z}$ such that $a - b = kn$. Multiplying both sides by $a + b$ yields

$$a^2 - b^2 = nk(a + b).$$

   Let $\ell = k(a + b)$; observe that $\ell \in \mathbb{Z}$. Since $a^2 - b^2 = n\ell$, by definition of divides, $n \mid (a^2 - b^2)$ as required.