Name:

Math 346: Final exam practice

May 18, 2023

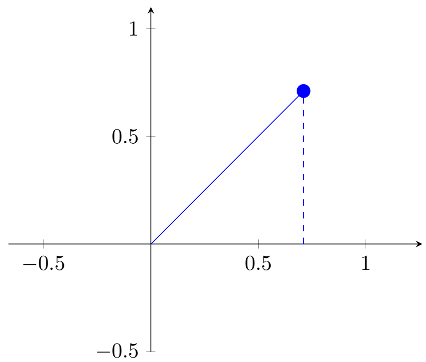
Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Avoid using the back of the page. There is a note sheet at the end, which you can tear out. Calculators are not allowed.

1. Short answer questions. You do not have to show your work.

   (a) (5 points) Let $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$. Compute z^{2023} in rectangular form.

   (a) ______________

   Solution: We first convert into exponential form:



   From the graph, we see that $z = e^{i\pi/4}$. Notice that

$$z^8 = (e^{i\pi/4})^8 = e^{2\pi i} = 1.$$

   But $2023 = 8 \cdot 252 + 7$, so

$$\begin{aligned} z^{2023} &= z^{8\cdot 252+7} \\ &= (z^8)^{252} \cdot z^7 \\ &= 1^{252} \cdot (e^{i\pi/4})^7 \\ &= e^{7i\pi/4}. \end{aligned}$$

   (b) (5 points) If $\overrightarrow{v} = (1,3)$ and $\overrightarrow{w} = (-2,4)$, what is $\text{proj}_{\overrightarrow{w}}(\overrightarrow{v})$?

   (b) ______________

   Solution: It is

$$\begin{aligned} \frac{\langle(-2,4),(1,3)\rangle}{\langle(-2,4),(-2,4)\rangle}(-2,4) &= \frac{-2\cdot 1 + 4\cdot 3}{(-2)^2+4^2}(-2,4) \\ &= \frac{10}{20}(-2,4) \\ &= (-1,2). \end{aligned}$$

(c) (5 points) Compute $\|1 + 3\sin 2x - \cos 5x\|^2$.

(c) ______________

Solution: When we expand out $\langle 1 + 3\sin 2x - \cos 5x, 1 + 3\sin 2x - \cos 5x\rangle$, all of the terms are 0, since the functions are orthogonal to each other, except for

$$\langle 1, 1\rangle + \langle 3\sin 2x, 3\sin 2x\rangle + \langle -\cos 5x, -\cos 5x\rangle = \langle 1, 1\rangle + 9\langle \sin 2x, \sin 2x\rangle + \langle \cos 5x, \cos 5x\rangle .$$

By Theorem 7.4 from my notes, the values of the three inner products above are 1, $\frac{1}{2}$, and $\frac{1}{2}$, respectively. Thus our answer is

$$1 + 9\frac{1}{2} + \frac{1}{2} = 6.$$

(d) (10 points) Given that $x^2 - 2$ is a solution to $y'' + xy' + y = 3x^2$ and x is a solution to $y'' + xy' + y = x^2 + x$, find a solution to $y'' + xy' + y = 2x^2 - x$.

(d) ______________

Solution: Let $L = D^2 + xD + 1$. From the above, we have

$$L(x^2 - 2) = 3x^2 \text{ and } L(x) = x^2 + x.$$

Observe that $2x^2 - x = 3x^2 - (x^2 + x)$. Since L is linear,

$$L((x^2 - 2) - x) = L(x^2 - 2) - L(x) = 3x^2 - (x^2 + x).$$

Therefore our solution is $x^2 - 2 - x$ (or better, $x^2 - x - 2$).

(e) (5 points) Give an example of a 2nd order differential equation which is neither linear nor separable.

Solution: There are lots of answers; $(y'')^2 = x + y$ is one example.

2. (15 points) Find area of the triangle with vertices at $(-1, 2), (3, 1)$, and $(2, 5)$.

Solution: Let $\vec{v}$ be the vector from $(-1, 2)$ to $(3, 1)$, and $\vec{w}$ the vector from $(-1, 2)$ to $(2, 5)$. We have $\vec{v} = (4, -1)$ and $\vec{w} = (3, 3)$. The area of the parallelogram defined by $\vec{v}$ and $\vec{w}$ is the absolute value of

$$\det \begin{bmatrix} 4 & -1 \\ 3 & 3 \end{bmatrix}$$

which equals $|4 \cdot 3 - (-1) \cdot 3| = 15$. The area of the triangle is half that of the parallelogram, so $\frac{15}{2}$.

3. Let $\vec{v}_1 = (3/5, 4/5)$ and $\vec{v}_2 = (4/5, -3/5)$.

(a) (15 points) Show that $\vec{v}_1, \vec{v}_2$ is an orthonormal list.

Solution: We have

$$\begin{aligned}
\langle \vec{v}_1, \vec{v}_2 \rangle &= \langle (3/5, 4/5), (4/5, -3/5) \rangle \\
&= \frac{3}{5} \cdot \frac{4}{5} - \frac{4}{5} \cdot \frac{3}{5} \\
&= 0, \\
\langle \vec{v}_1, \vec{v}_1 \rangle &= \langle (3/5, 4/5), (3/5, 4/5) \rangle \\
&= \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 \\
&= \frac{9}{25} + \frac{16}{25} \\
&= 1, \\
\langle \vec{v}_2, \vec{v}_2 \rangle &= \langle (4/5, -3/5), (4/5, -3/5) \rangle \\
&= \left(\frac{4}{5}\right)^2 + \left(-\frac{3}{5}\right)^2 \\
&= \frac{16}{25} + \frac{9}{25} \\
&= 1.
\end{aligned}$$

The claim follows.

(b) (10 points) Write the vector $(2, -3)$ in terms of $\vec{v}_1$ and $\vec{v}_2$.

Solution: Since $\vec{v}_1, \vec{v}_2$ is orthonormal, if we take

$$(2, -3) = c_1 \vec{v}_1 + c_2 \vec{v}_2,$$

then c_1, c_2 are given by the appropriate inner products. In particular,

$$\begin{aligned}
c_1 &= \langle \vec{v}_1, (2, -3) \rangle \\
&= 2 \cdot \frac{3}{5} - 3 \cdot \frac{4}{5} \\
&= \frac{6}{5} - \frac{12}{5} \\
&= -\frac{6}{5}, \\
c_2 &= \langle \vec{v}_2, (2, -3) \rangle \\
&= 2 \cdot \frac{4}{5} + 3 \cdot \frac{3}{5} \\
&= \frac{8}{5} + \frac{9}{5} \\
&= \frac{17}{5}.
\end{aligned}$$

Thus

$$(2, -3) = -\frac{6}{5} \vec{v}_1 + \frac{17}{5} \vec{v}_2.$$

4. Solve each of the following for z over the complex numbers. Put your answers in rectangular form.

(a) (15 points) $z^3 = i$

Solution: We have $z^3 = e^{i\pi/2}$. Since the magnitude on the right hand side is 1, the magnitude of z is also 1. The argument of z is either $\pi/6$, $5\pi/6$, or $3\pi/2$. (This follows since $3(\pi/6) = \pi/2$, and the roots are spaced evenly around the origin.) The roots are therefore

$$e^{i\cdot\pi/6}, e^{i\cdot5\pi/6}, e^{i\cdot3\pi/2}.$$

Applying Euler's formula, we conclude that the roots are

$$\frac{\sqrt{3}}{2} + \frac{1}{2}i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i, -i.$$

(b) (15 points) $z^2 = 2 + 2i\sqrt{3}$

Solution: We apply the same method as in the last problem. The right hand side is $4e^{i\cdot\pi/3}$, so the magnitude of z is 2. The argument of z is either $\pi/6$ or $7\pi/6$. The roots are therefore $2e^{i\pi/6}$ and $2e^{7i\pi/6}$, which respectively equal

$$\sqrt{3} + i, -\sqrt{3} - i.$$

5. Find exponential Fourier series for the following functions.

(a) (15 points) The 2π-periodic function $f(x)$ given by

$$f(x) = \begin{cases} 1 & \text{if } -\pi \le x < -\frac{\pi}{2} \\ 0 & \text{if } -\frac{\pi}{2} \le x < 0 \\ 1 & \text{if } 0 \le x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \le x < \pi. \end{cases}$$

Solution: Observe that $f(x)$ is just the square wave with twice the frequency—geometrically, the graph is squashed horizontally by a factor of 2. Thus $f(x) = s(2x)$. So to get the Fourier series, we substitute $2x$ for x into the Fourier series of the square wave, and obtain

$$\frac{1}{2} + \frac{1}{\pi i} \sum_{k=-\infty}^{\infty} \frac{1}{2k-1} e^{(4k-2)ix}.$$

(b) (15 points) The 2π-periodic function $g(x)$ given by

$$g(x) = \begin{cases} 0 & \text{if } -\pi \le x < -\frac{\pi}{2} \\ 1 & \text{if } -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \le x < \pi. \end{cases}$$

Solution: This is the square wave shifted left by $\frac{\pi}{2}$, so $g(x) = s\left(x + \frac{\pi}{2}\right)$. Substituting $x + \frac{\pi}{2}$ into the Fourier series for the square wave, we obtain

$$\frac{1}{2} + \frac{1}{\pi i} \sum_{k=-\infty}^{\infty} \frac{1}{2k-1} e^{(2k-1)i(x+\pi/2)}.$$

Unfortunately, this is no longer a proper Fourier series, but the simplification is not terrible. We have

$$\begin{aligned}
e^{(2k-1)i(x+\pi/2)} &= e^{(2k-1)ix+i(2k-1)\pi/2} \\
&= e^{(2k-1)ix}e^{i(2k-1)\pi/2} \\
&= e^{(2k-1)ix}\left(e^{i\pi/2}\right)^{(2k-1)} \\
&= e^{(2k-1)ix}i^{2k-1}.
\end{aligned}$$

Thus the series can be rewritten

$$\frac{1}{2}+\frac{1}{\pi i}\sum_{k=-\infty}^{\infty}\frac{i^{2k-1}}{2k-1}e^{(2k-1)ix}.$$

Note that $i^{2k-1}=i^{2k-2}i=(i^2)^{k-1}i=(-1)^{k-1}i$, in which case the series becomes

$$\frac{1}{2}+\frac{1}{\pi}\sum_{k=-\infty}^{\infty}\frac{(-1)^{k-1}}{2k-1}e^{(2k-1)ix};$$

but it is unnecessary to make this simplification.

6. (15 points) The function $f(x)$ is given by

$$f(x)=\begin{cases}1 & \text{if } 0\le x\le 2 \\ 0 & \text{otherwise}\end{cases}.$$

Compute its Fourier transform $\hat{f}$.

Solution: We have

$$\begin{aligned}
\hat{f}(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\alpha x}f(x)\,\mathrm{dx} \\
&= \frac{1}{2\pi}\int_{0}^{2} e^{-i\alpha x}\,\mathrm{dx}.
\end{aligned}$$

When $\alpha\neq 0$, we get

$$\begin{aligned}
\hat{f}(\alpha) &= \frac{1}{-2\pi i\alpha}e^{-i\alpha x}\Big|_0^2 \\
&= \frac{e^{-2i\alpha}-1}{-2\pi i\alpha} \\
&= \frac{1-e^{-2i\alpha}}{2\pi i\alpha}.
\end{aligned}$$

When $\alpha=0$, we get

$$\begin{aligned}
\hat{f}(\alpha) &= \frac{1}{2\pi}\int_0^2 1\,\mathrm{dx} \\
&= \frac{1}{\pi}.
\end{aligned}$$

Note that if we use L'Hôpital's rule to evaluate $\lim_{\alpha \to 0} \dfrac{1 - e^{-2i\alpha}}{2\pi i \alpha}$, we get $\frac{1}{\pi}$, so $\hat{f}(\alpha)$ is continuous. (You didn't have to check this, but it's an expected conclusion.)

7. Solve the following differential equations.
   (a) (15 points) $y' + 2xy = x$.

Solution: The integrating factor is $e^{\int 2x\, \mathrm{dx}} = e^{x^2}$. Thus

$$\frac{d}{dx}(e^{x^2} y) = x e^{x^2}.$$

We therefore have

$$e^{x^2} y = \int x e^{x^2}\, \mathrm{dx}.$$

Using $u = x^2$ on the right hand side, we get

$$e^{x^2} y = \frac{1}{2} e^{x^2} + C.$$

Therefore

$$y = \frac{1}{2} + C e^{-x^2}.$$

   (b) (5 points) $y' + 2xy = -3x$.

Solution: Let $L = D + 2x$. From part (a), we have

$$L\left(\frac{1}{2} + Ce^{-x^2}\right) = x.$$

Therefore

$$L\left(-3 \cdot \left(\frac{1}{2} + Ce^{-x^2}\right)\right) = -3x$$

since L is linear. Letting $D = -3C$, we get our solution as

$$-\frac{3}{2} + De^{-x^2}.$$

8. Solve the following differential equations.
   (a) (15 points) $y'' - 2y' - 15y = 0$.

Solution: The differential operator is $D^2 - 2D - 15$, which factors as $(D-5)(D+3)$. A solution of $(D-5)(y) = 0$ is e^{5x}, while a solution of $(D+3)(y) = 0$ is e^{-3x}. Since $5 \neq -3$, the two solutions are linearly indepedent. It follows that the general solution is

$$y = c_1 e^{5x} + c_2 e^{-3x}.$$

   (b) (5 points) $y'' - 2y' - 15y = 5$.

Solution: By inspection, $y_0 = -\frac{1}{3}$ is a solution, as

$$\left(-\frac{1}{3}\right)'' - 2\left(-\frac{1}{3}\right)' - 15\left(-\frac{1}{3}\right) = 0 - 0 + 5 = 5.$$

The associated homogeneous equation was the previous part, so the final answer is

$$-\frac{1}{3} + c_1 e^{5x} + c_2 e^{-3x}.$$

You may use the following facts.

- The trigonometric Fourier series for the square wave is

$$\frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin(2k-1)x.$$

- The exponential Fourier series for the square wave is

$$\frac{1}{2} + \frac{1}{\pi i} \sum_{k=-\infty}^{\infty} \frac{1}{2k-1} e^{(2k-1)ix}.$$

- If $g(x) = f(x-a)$ and $h(x) = f(cx)$, then

$$\hat{g}(\alpha) = e^{-i\alpha a} \hat{f}(\alpha) \quad \text{and} \quad \hat{h}(\alpha) = \frac{1}{|c|} \hat{f}\left(\frac{\alpha}{c}\right).$$