Name:

Math 346: Practice Exam 3

April 23, 2023

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Avoid using the back of the page. There is a note sheet at the end, which you can tear out. Calculators are not allowed.

1. Short answer questions. You do not have to show your work.

   (a) (5 points) Let $f(x) = 5 - 3e^{ix} + e^{-2ix} + 4e^{3ix}$. In $L^2[-\pi, \pi]$, calculate $\langle \cos 3x, f(x) \rangle$.

   (a) ______________

   Solution: The inner product of $\cos 3x$ with each term of $f(x)$ is zero, except for the last term, where we get

   $$\begin{aligned} \langle \cos 3x, 4e^{3ix} \rangle &= \langle \cos 3x, 4(\cos 3x + i \sin 3x) \rangle \\ &= 4 \langle \cos 3x, \cos 3x \rangle \\ &= 4/2 = 2. \end{aligned}$$

   (b) (5 points) Compute $\|e^{ix} - 2e^{2ix} + e^{-3ix}\|^2$.

   (b) ______________

   Solution: When we expand out

   $$\langle e^{ix} - 2e^{2ix} + e^{-3ix}, e^{ix} - 2e^{2ix} + e^{-3ix} \rangle,$$

   all of the terms yield zero except

   $$\langle e^{ix}, e^{ix} \rangle + 4 \langle e^{2ix}, e^{2ix} \rangle + \langle e^{-3ix}, e^{-3ix} \rangle.$$

   Each inner product above is 1, hence the answer is $1 + 4 + 1 = 6$.

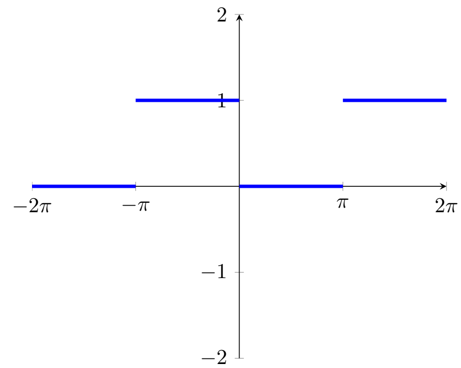
   (c) (5 points) Compute $\|1 - \cos 2x + 5 \sin 3x\|^2$.

   (c) ______________

   Solution: This is similar, but recall that $\|\cos 2x\|^2 = \|\sin 3x\|^2 = \frac{1}{2}$, so those terms get a $\frac{1}{2}$. The cross terms all have inner product zero. We therefore get

   $$\begin{aligned} 1^2 + \frac{1}{2}1^2 + \frac{1}{2}5^2 &= 1 + \frac{1}{2} + \frac{25}{2} \\ &= 14. \end{aligned}$$

   (d) (5 points) Find the trigonometric Fourier series for the 2π-periodic function given below.



Solution: There are a few ways to relate this function to the square wave, but the easiest is to write $f(x) = 1 - s(x)$, and so the Fourier series becomes

$$\frac{1}{2} - \frac{2}{\pi}\sum_{k=1} \frac{\sin(2k-1)x}{2k-1}.$$

(e) (5 points) State the Fourier inversion formula.

Solution: It is $\hat{\hat{f}}(x) = \frac{1}{2\pi} f(-x)$, or alternatively

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{i\alpha x}\, d\alpha.$$

2. (a) (15 points) Let $f(x)$ be the 2π-periodic function given by $f(x) = \dfrac{x}{\pi}$ on $[-\pi, \pi]$. Compute its exponential Fourier series.

Solution: Moving the $1/\pi$ to the outside of the integral, we have

$$c_k = \langle e^{ikx}, x/\pi \rangle = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} x e^{-ikx}\, \mathrm{dx}.$$

If $k = 0$, the integrand is odd, so the integral is 0. If $k \neq 0$, we use integration by parts. Let $u = x$ and $dv = e^{-ikx}\,\mathrm{dx}$. Then $du = dx$ and $v = \frac{1}{-ik} e^{-ikx}$, yielding

$$\begin{aligned} c_k &= \frac{1}{2\pi^2}\left(-\frac{1}{ik} x e^{-ikx} \Big|_{-\pi}^{\pi} + \frac{1}{ik}\int_{-\pi}^{\pi} e^{-ikx}\,\mathrm{dx} \right) \\ &= \frac{1}{2\pi^2}\left(-\frac{1}{ik}\pi e^{-ik\pi} - \frac{1}{ik}\pi e^{ik\pi} + \frac{1}{-i^2k^2} e^{-ik\pi} - \frac{1}{-i^2k^2} e^{ik\pi} \right). \end{aligned}$$

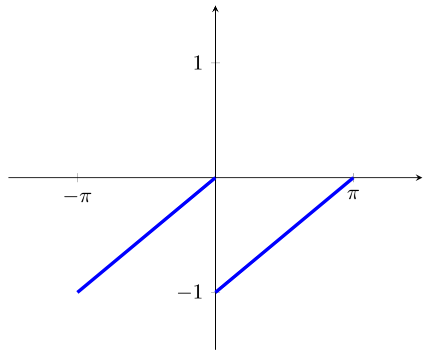
Since $e^{i\pi} = -1$, the above simplifies to

$$\frac{(-1)^{k+1}}{\pi i k}.$$

Thus, the Fourier series is

$$\frac{1}{i\pi}\sum_{k\neq 0} \frac{(-1)^{k+1}}{k} e^{ikx}.$$

(b) (15 points) Let $g(x)$ be the 2π-periodic function given by the following graph on $[-\pi, \pi]$. Compute its exponential Fourier series.



Solution: We have $g(x) = f(x) - s(x)$, and so the Fourier series is

$$\frac{1}{i\pi}\sum_{k\neq 0}\frac{(-1)^{k+1}}{k}e^{ikx} - \left(\frac{1}{2} + \frac{1}{\pi i}\sum_{\ell=-\infty}^{\infty}\frac{1}{2\ell+1}e^{(2\ell+1)ix}\right)$$

It is easier to simplify if we deal with odd and even terms separately. For even terms, $k \neq 0$, we get

$$-\frac{1}{i\pi k}e^{ikx}.$$

For odd terms, setting $k = 2\ell + 1$, we get

$$\frac{1}{i\pi(2\ell+1)}e^{i(2\ell+1)x} - \frac{1}{\pi i(2\ell+1)}e^{i(2\ell+1)x}.$$

These cancel, so we only have the even terms.

Thus the Fourier series is

$$-\frac{1}{2} + \frac{1}{\pi i}\sum_{\substack{k \text{ even} \\ k\neq 0}}\frac{1}{k}e^{ikx}.$$

3. (15 points) Let $f(x) = 3\cos x - \sin 2x + 2\cos 2x$. Compute the exponential Fourier series of f.

Solution: We have

$$\begin{aligned} 3\cos x - \sin 2x + 2\cos 2x &= \frac{3}{2}(e^{ix} + e^{-ix}) - \frac{1}{2i}(e^{2ix} - e^{-2ix}) + e^{2ix} + e^{-2ix} \\ &= \left(1 + \frac{1}{2}i\right)e^{-2ix} + \frac{3}{2}e^{-ix} + \frac{3}{2}e^{ix} + \left(1 - \frac{1}{2}i\right)e^{2ix}. \end{aligned}$$

This is already an exponential Fourier series, so we're done.

4. (15 points) Let $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$. Suppose that f is an even function. Find an identity satisfied by the c_k, and prove your answer.

Solution: We have $f(x) = f(-x)$, so

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = \sum_{k=-\infty}^{\infty} c_k e^{-ikx}.$$

Replacing k with $-k$ in the second summation, we get

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = \sum_{k=-\infty}^{\infty} c_{-k} e^{ikx}.$$

It follows that $c_k = c_{-k}$.

5. (10 points) The function $f(x)$ is given by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}.$$

Compute its Fourier transform $\hat{f}$.

Solution: We have

$$\begin{aligned} \hat{f}(\alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha x} f(x)\, \mathrm{dx} \\ &= \frac{1}{2\pi} \int_{0}^{2} e^{-i\alpha x}\, \mathrm{dx}. \end{aligned}$$

When $\alpha \neq 0$, we get

$$\begin{aligned} \hat{f}(\alpha) &= \frac{1}{-2\pi i\alpha} e^{-i\alpha x}\Big|_0^2 \\ &= \frac{e^{-2i\alpha} - 1}{-2\pi i\alpha} \\ &= \frac{1 - e^{-2i\alpha}}{2\pi i\alpha}. \end{aligned}$$

When $\alpha = 0$, we get

$$\begin{aligned} \hat{f}(\alpha) &= \frac{1}{2\pi} \int_0^2 1\, \mathrm{dx} \\ &= \frac{1}{\pi}. \end{aligned}$$

Note that if we use L'Hôpital's rule to evaluate $\lim\limits_{\alpha \to 0} \dfrac{1 - e^{-2i\alpha}}{2\pi i\alpha}$, we get $\frac{1}{\pi}$, so $\hat{f}(\alpha)$ is continuous. (You didn't have to check this, but it's an expected conclusion.)