Name:

Math 346: Exam 2

March 15, 2023 (roughly)

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Avoid using the back of the page; instead, there is an extra sheet at the end that you can use. Calculators are not allowed.

1. In $\mathbb{R}^3$, let

$$\overrightarrow{u} = (2, -1, 1) \text{ and } \overrightarrow{w} = (3, 2, -1).$$

Compute each of the following.

(a) (5 points) $2\overrightarrow{u} - \overrightarrow{w}$

Solution:

$$(2 \cdot 2 - 3, 2 \cdot (-1) - 2, 2 \cdot 1 - (-1)) = (1, -4, 3).$$

(a) ______________

(b) (5 points) $\langle \overrightarrow{u}, \overrightarrow{w} \rangle$

Solution:

$$2 \cdot 3 + (-1) \cdot 2 + 1 \cdot (-1) = 3.$$

(b) ______________

(c) (5 points) $\|\overrightarrow{u}\|^2$

Solution:

$$\begin{aligned} \|\overrightarrow{u}\|^2 &= \langle \overrightarrow{u}, \overrightarrow{u} \rangle \\ &= 2^2 + (-1)^2 + 1^2 \\ &= 6. \end{aligned}$$

(c) ______________

2. (a) (15 points) Find the area of the triangle with vertices at $(2, -1)$, $(1, 4)$, and $(5, 2)$.

Solution: Label our points, in order, P, Q, R. Then the triangle is defined by the 2 vectors $\overrightarrow{PQ} = (1, 4) - (2, -1) = (-1, 5)$ and $\overrightarrow{PR} = (5, 2) - (2, -1) = (3, 3)$. Our triangle has half the area of the parallelogram defined by these two vectors, and hence it is

$$\begin{aligned} \frac{1}{2} \left| \det \begin{bmatrix} -1 & 5 \\ 3 & 3 \end{bmatrix} \right| &= \frac{1}{2} |-3 - 15| \\ &= 9. \end{aligned}$$

(b) (5 points) Compute $\det \left[\begin{smallmatrix} -2 & 5 \\ 3 & 2 \end{smallmatrix}\right]$.

> **Solution:** It is $-4 - 15 = -19$.

(c) (10 points) Given a 3×3 matrix A, a new matrix B is obtained from A by multiplying all of the entries of A by 3. Write an equation relating $\det B$ to $\det A$, and justify that the equation is correct.

> **Solution:** We have $\det B = 27 \det A$. The reason is that multiplying A by 3 is the same as multiplying each row of A by 3. Each row multiplication also multiplies the determinant by 3. Since there are 3 rows, the total multiplication of the determinant is $3^3 = 27$.

3. (a) (15 points) Determine if the list

$$(1, 1, -3), (2, -3, 4), (7, -3, -1)$$

is linearly dependent or linearly independent. If it is dependent, eliminate the fewest number of vectors possible to obtain a linearly independent set. Make sure to justify that the resulting set is linearly independent!

> **Solution:** We set up the matrix
>
> $$\begin{bmatrix} 1 & 2 & 7 \\ 1 & -3 & -3 \\ -3 & 4 & -1 \end{bmatrix}$$
>
> and row reduce. It turns out that the rref is
>
> $$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$
>
> Thus the system is linearly dependent. The 3rd column is free, so we can omit it. The remaining pair $(1, 1, -3), (2, -3, 4)$ must form a linearly independent set.

(b) (5 points) Compute

$$\det \begin{bmatrix} 1 & 2 & 7 \\ 1 & -3 & -3 \\ -3 & 4 & -1 \end{bmatrix}.$$

> **Solution:** From the previous problem, this matrix has rank 2, and hence its determinant is 0.

(c) (10 points) Determine if the list of functions $e^{ix}, \cos x, \sin x$ is linearly dependent or independent. Justify your answer.

> **Solution:** One can compute a Wronskian, but it is easier to observe that
>
> $$e^{ix} = \cos x + i \sin x$$
>
> which shows the list is linearly dependent.

4. Consider the vectors

$$\vec{b}_1 = (1/3, 2/3, 2/3) \quad \vec{b}_2 = (-2/3, -1/3, 2/3) \quad \vec{b}_3 = (2/3, -2/3, 1/3).$$

(a) (20 points) Show that $\vec{b}_1, \vec{b}_2, \vec{b}_3$ forms an orthonormal list.

Solution: We have

$$\begin{aligned}\|\vec{b}_1\|^2 &= \frac{1}{3^2} + \frac{2^2}{3^2} + \frac{2^2}{3^2} \\ &= \frac{1}{9} + \frac{4}{9} + \frac{4}{9} \\ &= 1.\end{aligned}$$

Therefore $\vec{b}_1$ is a unit vector. Similar calculations hold for $\vec{b}_2$ and $\vec{b}_3$. Furthermore,

$$\begin{aligned}\langle \vec{b}_1, \vec{b}_2 \rangle &= \frac{1 \cdot (-2)}{9} + \frac{2 \cdot (-1)}{9} + \frac{2 \cdot 2}{9} \\ &= \frac{-2 - 2 + 4}{9} \\ &= 0.\end{aligned}$$

Therefore $\vec{b}_1$ and $\vec{b}_2$ are orthogonal. One should check $\langle \vec{b}_1, \vec{b}_3 \rangle = 0$ and $\langle \vec{b}_2, \vec{b}_3 \rangle = 0$ in a similar way; I leave the details to the reader.

(b) (10 points) Write the vector $(1, \frac{1}{2}, -3)$ in terms of the vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$.

Solution: Label the last 3 vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$, respectively. The $\vec{v}_i$ are identical to those given in one of the homework problems; as in that problem, the three vectors form an orthonormal basis for $\mathbb{R}^3$. Therefore if $(1, \frac{1}{2}, -3) = \sum c_i \vec{v}_i$, we have $c_i = (1, \frac{1}{2}, -3) \cdot \vec{v}_i$. In other words

$$\begin{aligned}c_1 &= (1, 1/2, -3) \cdot (1/3, 2/3, 2/3) = -4/3 \\ c_2 &= (1, 1/2, -3) \cdot (-2/3, -1/3, 2/3) = -17/6 \\ c_3 &= (1, 1/2, -3) \cdot (2/3, -2/3, 1/3) = -2/3.\end{aligned}$$

Therefore

$$\left(1, \frac{1}{2}, -3\right) = -\frac{4}{3}\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) - \frac{17}{6}\left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) - \frac{2}{3}\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right).$$

5. (15 points) Using the inner product $\langle f, g \rangle = \frac{1}{2}\int_{-1}^{1} \overline{f} g \, dx$, find the projection of $x + 2$ onto x^2.

Solution: We have $\text{proj}_{x^2}(x+2) = \frac{\langle x^2, x+2 \rangle}{\langle x^2, x^2 \rangle} x^2$. We calculate the inner products:

$$\begin{aligned}\langle x^2, x+2 \rangle &= \frac{1}{2}\int_{-1}^{1} x^3 + 2x^2 \, dx \\ &= \frac{x^4}{8} + \frac{2x^3}{6}\bigg|_{-1}^{1} \\ &= \frac{2}{3}.\end{aligned}$$

Also,

$$\begin{aligned}\langle x^2, x^2 \rangle &= \frac{1}{2}\int_{-1}^{1} x^4\,dx \\ &= \frac{x^5}{10}\bigg|_{-1}^{1} \\ &= \frac{1}{5}.\end{aligned}$$

Therefore the projection is

$$\frac{2/3}{1/5}x^2 = \frac{10}{3}x^2.$$