Name:

Math 346: Exam 1

February 6, 2023

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Avoid using the back of the page; instead, there is an extra sheet at the end that you can use. Calculators are not allowed.

1. Let $z = 1 + i$ and $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

   (a) (5 points) Compute $\frac{w+\overline{w}}{2}$.

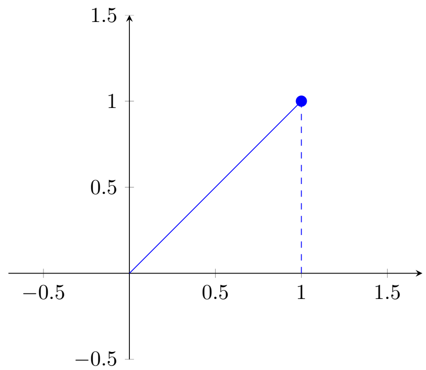
   (a) ____________

   Solution: This can be done directly, or by observing that $\frac{w+\overline{w}}{2} = \Re(w) = \frac{1}{2}$.

   (b) (5 points) Write z in exponential form.

   (b) ____________

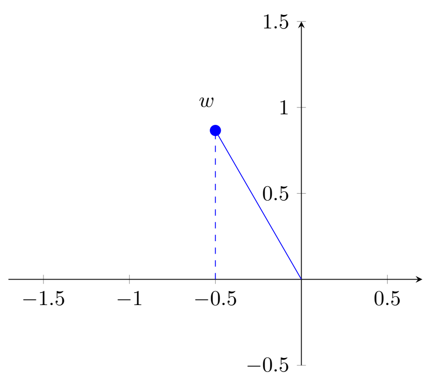
   Solution: Graphing z, we see that $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$. Thus we have $z = \sqrt{2}e^{i\pi/4}$.



   (c) (10 points) Compute w^{50}.

   (c) ____________

   Solution: Converting w to exponential form, we have $r = 1$ and $\theta = 2\pi/3$, so $w = e^{2\pi i/3}$. Then $w^{50} = e^{100\pi i/3} = e^{4\pi i/3}$ (by subtracting off enough multiples of 2π).



(d) (10 points) Compute w/z in the form of your choice.

(d) _______________

Solution: Exponential is easiest; from the previous two parts, $w = e^{2\pi i/3}$ and $z = \sqrt{2}e^{i\pi/4}$, so

$$\begin{aligned} \frac{w}{z} &= \frac{e^{2\pi i/3}}{\sqrt{2}e^{i\pi/4}} \\ &= \frac{1}{\sqrt{2}}e^{\frac{2\pi}{3}i - \frac{\pi}{4}i} \\ &= \frac{1}{\sqrt{2}}e^{5\pi i/12}. \end{aligned}$$

2. Evaluate the following. Write your answer in the form specified.

(a) (10 points) $e^{-2\pi i} - 3e^{-4\pi i} - 2e^{-6\pi i}$, rectangular

(a) _______________

Solution: We have $e^{-2\pi i} = e^{-4\pi i} = e^{-6\pi i} = 1$, so the answer is $1 - 3 - 2 = -4$.

(b) (10 points) $-e^{3\pi i/2} + 2e^{5\pi i/6}$, rectangular

(b) _______________

Solution: Converting each to rectangular (I omit the graphs), we get

$$e^{3\pi i/2} = -i \text{ and } 2e^{5\pi i/6} = -\sqrt{3} + i.$$

Thus the expression above is

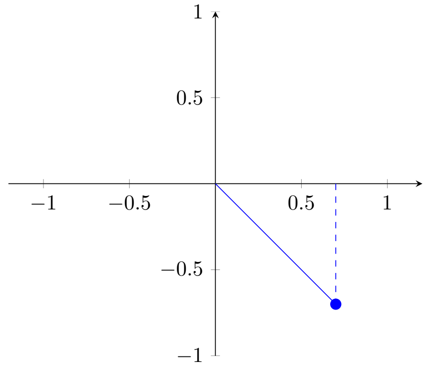
$$-(-i) + (-\sqrt{3} + i) = -\sqrt{3} + 2i.$$

(c) (10 points) $\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) \cdot (-i)$, exponential

(c) _______________

Solution: We can either multiply first and then convert to exponential, or convert first and then multiply. Let's multiply first:

$$\begin{aligned} \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) \cdot (-i) &= -\frac{i}{\sqrt{2}} - \frac{i^2}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i. \end{aligned}$$



Graphing, we get the exponential form $e^{-i\pi/4}$.

3. Solve each of the following for z over the complex numbers.

(a) (10 points) $z^2 - z + 1 = 0$

Solution: Applying the quadratic formula, we obtain

$$z = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 1}}{2} = \frac{1}{2} \pm \frac{\sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

(b) (10 points) $z^3 = i$

Solution: Let $z = re^{i\theta}$. Observe that $i = e^{i\pi/2} = e^{i5\pi/2} = e^{i9\pi/2}$. Substituting, we obtain

$$r^3 e^{i\cdot 3\theta} = \begin{cases} e^{i\pi/2} & \text{or} \\ e^{i5\pi/2} & \text{or} \\ e^{i9\pi/2}. \end{cases}$$

In every case, $r^3 = 1$ so $r = 1$. As for θ, we get $3\theta = \pi/2$ or $5\pi/2$ or $9\pi/2$, so $\theta = \pi/6$ or $5\pi/6$ or $3\pi/2$. Thus our answers are

$$z = e^{\pi i/6} \text{ or } e^{5\pi i/6} \text{ or } e^{3\pi i/2}.$$

4. (15 points) Let $z = x + iy$. Show that $\Im(z) = \frac{z - \overline{z}}{2i}$.

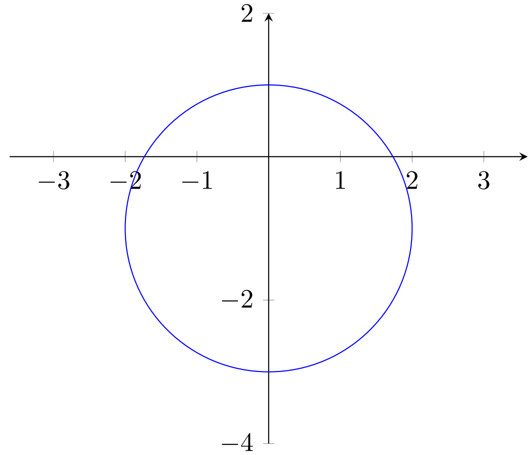
Solution: We have $\overline{z} = x - iy$, so

$$\begin{aligned} \frac{z - \overline{z}}{2i} &= \frac{(x+iy) - (x-iy)}{2i} \\ &= \frac{x + iy - x + iy}{2i} \\ &= \frac{2iy}{2i} \\ &= y \\ &= \Im(z). \end{aligned}$$

5. Graph the following. Make sure to justify your answers.

(a) (10 points) $|z+i|=2$

Solution: Rewriting $|z+i| = |z-(-i)|$, this is the set of points 2 units from $-i$, which is a circle of radius 2 centered at $-i$.



(b) (10 points) $z-\overline{z}=2i$

Solution: Rewriting as $\frac{z-\overline{z}}{2i}=1$, by problem 4, this is the same as $\Im(z)=1$, which is a horizontal line through i.