Homework 7 selected solutions
Due: Friday, October 31

7.5.2 We have

$$\begin{aligned} a_0 &= \langle 1, f \rangle \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f \, dx \\ &= \frac{1}{4} \end{aligned}$$

since the region under the curve is a rectangle with area $\frac{\pi}{2}$. Next, for $n \geq 1$ we have

$$\begin{aligned} a_n &= \langle \cos nx, f \rangle / (1/2) \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ &= \frac{1}{\pi}\int_{0}^{\pi/2} \cos nx \, dx \\ &= \frac{1}{n\pi} \sin nx \Big]_0^{\pi/2}. \end{aligned}$$

The lower bound evaluates to 0. When we plug the upper bound into $\sin nx$, we get 0 when n is even, $+1$ when n is one more than a multiple of 4, and -1 when n is 3 more than a multiple of 4. The most compact way of writing this is that $a_n = 0$ when n is even, otherwise setting $n = 2k - 1$, we get

$$a_{2k-1} = \frac{(-1)^{k+1}}{(2k-1)\pi}.$$

For the bs, we get

$$\begin{aligned} b_m &= \langle \sin mx, f \rangle (1/2) \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx \, dx \\ &= \frac{1}{\pi}\int_{0}^{\pi/2} \sin mx \, dx \\ &= -\frac{1}{m\pi} \cos mx \Big]_0^{\pi/2}. \end{aligned}$$

The lower bound evaluates to $-\frac{1}{m\pi}$ (which we will subtract from the value when we plug in the upper bound). The upper bound, when plugged into $\cos mx$, yields 0 when m is odd, $+1$ when m is a multiple of 4, and -1 when m is 2 more than a multiple of 4. Putting this all together, we get that

$$b_m = \begin{cases} \frac{1}{m\pi} & \text{when } m \text{ is odd} \\ 0 & \text{when } m \text{ is a multiple of 4} \\ \frac{2}{m\pi} & \text{when } m \text{ is 2 more than a multiple of 4.} \end{cases}$$

One way of writing this is

$$\begin{aligned} F(f) = \frac{1}{4} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}\cos(2k-1)x + \frac{1}{\pi}\sum_{\ell=1}^{\infty}\frac{1}{2\ell-1}\sin(2\ell-1)x \\ + \frac{1}{\pi}\sum_{j=1}^{\infty}\frac{1}{4j-2}\sin(4j-2)x. \end{aligned}$$

7.5.3 Write f_n for the function in problem n. Note that $f_3(x) = s(x) - f_2(x)$. Subtracting Fourier series, we see that the $1/4$ terms cancel, as do all of the $\sin mx$ terms with odd m all cancel. The cosine terms and the $\sin mx$ terms with m of the form $4k-2$ collect; up to sign, the coefficients double. When we simplify, we obtain

$$-\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\cos[(2k-1)x] - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k-2}\sin[(4k-2)x].$$

7.5.6 Notice this is looks like the square wave, but with twice the frequency. This means that $f(x) = s(2x)$. Therefore

$$\begin{aligned} F(f) &= F(s)(2x) \\ &= \frac{1}{2} - \frac{2}{\pi}\sum_{k=1}\frac{1}{2k-1}\sin(4k-2)x. \end{aligned}$$