

Name: _____

Math 346: Exam 3a

November 18, 2025

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Avoid using the back of the page. There is a note sheet at the end, which you can tear out. Calculators are not allowed.

1. Short answer questions. All inner products are in $L^2[-\pi, \pi]$. You do not have to show your work.

(a) (5 points) Compute $\langle e^{-ix} - e^{ix} + 2e^{5ix}, 1 + 3e^{ix} - 5e^{2ix} + 7e^{3ix} \rangle$.

(a) _____

Solution: Every summand in the first vector is orthogonal to every summand in the second, with the exception of the pair $-e^{ix}, 3e^{ix}$. For that, we have

$$\begin{aligned}\langle -e^{ix}, 3e^{ix} \rangle &= -3 \langle e^{ix}, e^{ix} \rangle \\ &= -3.\end{aligned}$$

(b) (5 points) Compute $\|\cos 2x - 3 \sin 5x\|^2$.

(b) _____

Solution: This is

$$\begin{aligned}\langle \cos 2x - 3 \sin 5x, \cos 2x - 3 \sin 5x \rangle &= \langle \cos 2x, \cos 2x \rangle - 3 \langle \cos 2x, \sin 5x \rangle - 3 \langle \sin 5x, \cos 2x \rangle + 9 \langle \sin 5x, \sin 5x \rangle \\ &= \frac{1}{2} - 3 \cdot 0 - 3 \cdot 0 + 9 \cdot \frac{1}{2} \\ &= 5.\end{aligned}$$

(c) (5 points) Let $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$. In the exponential Fourier series of f , what is the value of c_{-5} ?

(c) _____

Solution: Note that c_{-5} is the coefficient of e^{-5ix} . Since

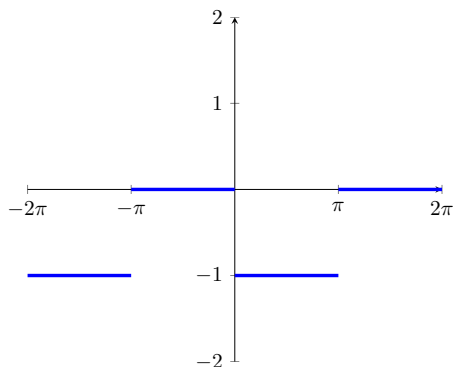
$$\cos nx = \frac{1}{2}e^{inx} + \frac{1}{2}e^{-inx},$$

the only term that will contribute is the $n = 5$ term, which is

$$\frac{1}{5^2} \cos 5x = \frac{1}{50}e^{5ix} + \frac{1}{50}e^{-5ix}.$$

Thus the desired coefficient is $\frac{1}{50}$.

- (d) (5 points) Find the exponential Fourier series for the 2π -periodic function given below.



Solution: We have $f(x) = -s(x)$, and so the Fourier series becomes

$$-\frac{1}{2} - \frac{1}{\pi i} \sum_{k=-\infty}^{\infty} \frac{1}{2k-1} e^{(2k-1)ix}.$$

2. (15 points) Let $f(x) = 3 \cos x - \sin 2x + 2 \cos 2x$. Compute the exponential Fourier series of f .

Solution: We have

$$\begin{aligned} 3 \cos x - \sin 2x + 2 \cos 2x &= \frac{3}{2}(e^{ix} + e^{-ix}) - \frac{1}{2i}(e^{2ix} - e^{-2ix}) + e^{2ix} + e^{-2ix} \\ &= \left(1 + \frac{1}{2}i\right)e^{-2ix} + \frac{3}{2}e^{-ix} + \frac{3}{2}e^{ix} + \left(1 - \frac{1}{2}i\right)e^{2ix}. \end{aligned}$$

This is already an exponential Fourier series, so we're done.

3. (a) (15 points) Let $f(x)$ be the 2π -periodic function given by $f(x) = x$ on $[-\pi, \pi]$. Compute its trigonometric Fourier series.

Solution: As f is an odd function, $a_n = 0$ for all n .

For the b 's, we have

$$\begin{aligned} b_m &= \frac{\langle \sin mx, f \rangle}{\langle \sin mx, \sin mx \rangle} \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin mx \, dx \\ &= -\frac{1}{m\pi} x \cos mx \Big|_{-\pi}^{\pi} + \frac{1}{m\pi} \int_{-\pi}^{\pi} \cos mx \, dx, \end{aligned}$$

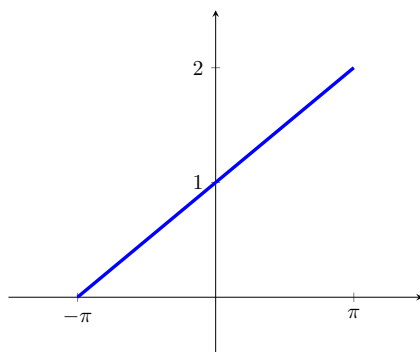
where the last line uses integration by parts with $u = x$ and $dv = \sin mx \, dx$. The last integral is $\frac{2}{m} \langle 1, \cos mx \rangle$, which is 0 since 1 and $\cos mx$ are orthogonal. Observe that $\cos m\pi = \cos -m\pi = (-1)^m$. Therefore the first term becomes

$$-\frac{1}{m\pi} (\pi(-1)^m - (-\pi)(-1)^m) = \frac{2}{m} (-1)^{m+1}.$$

The Fourier series is therefore

$$2 \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sin mx.$$

- (b) (15 points) Let $g(x)$ be the 2π -periodic function given by the following graph on $[-\pi, \pi]$. Compute its trigonometric Fourier series.



Solution: On $[-\pi, \pi]$, we have $g(x) = \frac{1}{\pi}x + 1$. This implies that

$$g(x) = \frac{1}{\pi}f(x) + 1,$$

and so the same identity holds for Fourier series:

$$\begin{aligned} F(g)(x) &= \frac{1}{\pi}F(f)(x) + 1 \\ &= \frac{1}{\pi} \left(2 \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sin mx \right) + 1 \\ &= 1 + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sin mx. \end{aligned}$$

4. (15 points) Let $f(x)$ be the 2π -periodic function defined on the interval $(-\pi, \pi]$ as follows:

$$f(x) = \begin{cases} 1 & \text{for } -\pi < x \leq -\frac{\pi}{2} \\ -1 & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 2 & \text{for } \frac{\pi}{2} \leq x \leq \pi. \end{cases}$$

Let $F(x)$ be the Fourier series for $f(x)$. Find the set of all x -values for which both of the following hold:

- $0 \leq x \leq 2\pi$, and
- $f(x) \neq F(x)$.

For each of these values, compute $F(x)$.

Solution: By Dirichlet's theorem, we need to check the places where f is discontinuous. On $[-\pi, \pi]$, these occur at $x = -\pi, -\frac{\pi}{2}, \frac{\pi}{2},$ and π . Since f is 2π -periodic, on $[0, 2\pi]$ these occur at $\frac{\pi}{2}, \pi,$ and $\frac{3\pi}{2}$. The value of F is the average of the left and right limits, yielding

$$\begin{aligned} F(\pi/2) &= \frac{1}{2} \\ F(\pi) &= \frac{3}{2} \\ F(3\pi/2) &= 0. \end{aligned}$$

5. (15 points) Let $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$. Suppose that f is an odd function. Find an identity satisfied by the c_k , and prove your answer.

Solution: We have $f(-x) = -f(x)$, so

$$\sum_{k=-\infty}^{\infty} c_k e^{-ikx} = - \sum_{k=-\infty}^{\infty} c_k e^{ikx}.$$

Replacing k with $-k$ in the first summation and moving the $-$ to the inside on the second, we get

$$\sum_{k=-\infty}^{\infty} c_{-k} e^{ikx} = \sum_{k=-\infty}^{\infty} (-c_k) e^{ikx}.$$

It follows that $c_{-k} = -c_k$.

6. (15 points) The function $f(x)$ is given by

$$f(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Compute its Fourier transform $\hat{f}$.

Solution: We have

$$\begin{aligned} \hat{f}(\alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha x} f(x) dx \\ &= \frac{1}{2\pi} \int_{-1}^0 e^{-i\alpha x} dx. \end{aligned}$$

When $\alpha \neq 0$, this equals

$$\begin{aligned} &= -\frac{1}{2\pi i\alpha} e^{-i\alpha x} \Big|_{-1}^0 \\ &= -\frac{e^0 - e^{i\alpha}}{2\pi i\alpha} \\ &= \frac{e^{i\alpha} - 1}{2\pi i\alpha} \end{aligned}$$

When $\alpha = 0$, we get

$$\begin{aligned} &= \frac{1}{2\pi} \int_{-1}^0 1 \, dx \\ &= \frac{1}{2\pi}. \end{aligned}$$

Therefore

$$\hat{f}(\alpha) = \begin{cases} \frac{e^{i\alpha} - 1}{2\pi i\alpha} & \alpha \neq 0 \\ \frac{1}{2\pi} & \alpha = 0. \end{cases}$$