Math 346: Exam 2b
October 19, 2022

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Calculators are not allowed.

1. In $\mathbb{C}^3$, let
   \[ \mathbf{u} = (2, -4, -1), \mathbf{v} = (2 - i, -1, 3i), \text{ and } \mathbf{w} = (1, 0, -3). \]

   Compute each of the following.
   
   (a) (5 points) $2\mathbf{u} - \mathbf{w}$

   Solution: $(3, -8, 1)$

   (b) (5 points) $\langle \mathbf{u}, \mathbf{v} \rangle$

   Solution:
   \[
   \langle \mathbf{u}, \mathbf{v} \rangle = 2 \cdot (2 - i) + (-4) \cdot (-1) + (-1) \cdot (3i)
   = 4 - 2i + 4 + 3i
   = 8 - 5i.
   \]

   (c) (5 points) $\langle \mathbf{v}, \mathbf{u} \rangle$

   Solution:
   \[
   \langle \mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle = 8 - 5i = 8 + 5i.
   \]

   (d) (5 points) $\|\mathbf{u}\|^2$

   Solution:
   \[
   \|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle
   = 2^2 + (-4)^2 + (-1)^2
   = 21.
   \]
(e) (5 points) \( \text{proj}_w(u) \)

**Solution:**

\[
\text{proj}_w(u) = \frac{\langle w, u \rangle}{\langle w, w \rangle} w = \frac{2 + 0 + 3}{1 + 0 + 9} (1, 0, -3) = \frac{1}{2} (1, 0, -3) = \left( \frac{1}{2}, 0, -\frac{3}{2} \right).
\]

(e) 

2. (a) (15 points) Consider the two lines with equations \( r = (1, 0, 1) + (2, 1, -1)t \) and \( r = (1, 0, 1) + (1, -2, 0)t \), respectively. Find a Cartesian equation for the plane which contains both lines.

**Solution:** We know that our normal vector \( n \) must be orthogonal to both \( (2, 1, -1) \) and \( (1, -2, 0) \). Letting \( n = (a, b, c) \), this means that

\[
2a + b - c = 0 \quad \text{and} \quad a - 2b + 0c = 0.
\]

We can solve this system using an augmented matrix; I omit the details. In any case, one of the solutions is \( (2, 1, 5) \). We know that \( (1, 0, 1) \) is a point on the line, so our equation is \( 2x + y + 5z = 2 \cdot 1 + 1 \cdot 0 + 5 \cdot 1 = 7 \), or in other words \( 2x + y + 5z = 7 \).

(b) (10 points) Find a parametric equation for the line which passes through the point \( (1, -3, 2) \) and is perpendicular to the plane with equation \( x - y + 2z = 5 \).

**Solution:** From the equation for the plane, the vector \( (1, -1, 2) \) is normal to the plane, and thus is parallel to the line. The vector equation is therefore

\[
r = (1, -1, 2)t + (1, -3, 2).
\]

3. (15 points) Find the area of the triangle with vertices at \( (1, 2) \), \( (2, 5) \), and \( (5, -1) \).

**Solution:** Label our points, in order, \( P, Q, R \). We have the vector \( PQ = (1, 3) \) and the vector \( PR = (4, -3) \). Since we are dealing with a triangle, which is half the area of the corresponding parallelogram, the area is half the absolute value of the determinant of

\[
\begin{vmatrix}
1 & 3 \\
4 & -3
\end{vmatrix}
\]

or \( \frac{1}{2} | -3 - 12 | = \frac{15}{2} \).
4. (a) (10 points) Compute \[ \begin{vmatrix} 4 & 3 & 0 \\ 1 & 2 & -1 \\ -1 & -2 & 2 \end{vmatrix} \].

**Solution:** If we add the second row to the third, we get
\[ \begin{vmatrix} 4 & 3 & 0 \\ 1 & 2 & -1 \\ 0 & 0 & 1 \end{vmatrix} \].

Proceeding by cofactors on the last row, we get \( 1 \cdot (8 - 3) = 5 \).

(b) (10 points) Given a \( 3 \times 3 \) matrix \( A \), a new matrix \( B \) is obtained from \( A \) by multiplying all of the entries of \( A \) by 3. Write an equation relating \( \det B \) to \( \det A \), and justify that the equation is correct.

**Solution:** We have \( \det B = 27 \det A \). The reason is that multiplying \( A \) by 3 is the same as multiplying each row of \( A \) by 3. Each row multiplication also multiplies the determinant by 3. Since there are 3 rows, the total multiplication of the determinant is \( (3)^3 = 27 \).

(c) (10 points) If we graph the solutions of the system of equations

\[
\begin{align*}
2x + y - 3z &= 0 \\
3x + 4y - 7z &= 0 \\
x - 2y + z &= 0 \\
x - 7y + 6z &= 0,
\end{align*}
\]

we get a line. Determine the rank of the matrix
\[ \begin{bmatrix} 2 & 1 & -3 \\ 3 & 4 & -7 \\ 1 & -2 & 1 \\ 1 & -7 & 6 \end{bmatrix} \].

**Solution:** The dimension of the solution set is 1, so there is 1 free variable. Therefore there are 2 pivot variables, and the rank is 2.

5. (15 points) Determine if the list

\[ (1, -3), (2, 0), (3, 3) \]

is linearly dependent or linearly independent. If it is dependent, eliminate the fewest number of vectors possible to obtain a linearly independent set. Make sure to justify that the resulting set is linearly independent!

**Solution:** Observe that \( 2(2, 0) - (1, -3) = (3, 3) \), so the list is linearly dependent. We may omit the last vector. The remaining vectors, \( (1, -3) \) and \( (2, 0) \), are not parallel, and hence form a linearly independent list.
6. Consider the vectors
\[ \mathbf{b}_1 = (1/3, 2/3, 2/3) \quad \mathbf{b}_2 = (-2/3, -1/3, 2/3) \quad \mathbf{b}_3 = (2/3, -2/3, 1/3). \]

(a) (15 points) Show that \( \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \) forms an orthonormal list.

**Solution:**

We have
\[
\| \mathbf{b}_1 \| = \frac{1}{3^2} + \frac{2^2}{3^2} + \frac{2^2}{3^2} = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1.
\]

Therefore \( \mathbf{b}_1 \) is a unit vector. Similar calculations hold for \( \mathbf{b}_2 \) and \( \mathbf{b}_3 \). Furthermore,
\[
\langle \mathbf{b}_1, \mathbf{b}_2 \rangle = \frac{1 \cdot (-2)}{9} + \frac{2 \cdot (-1)}{9} + \frac{2 \cdot 2}{9} = 0.
\]

Therefore \( \mathbf{b}_1 \) and \( \mathbf{b}_2 \) are orthogonal. One should check \( \langle \mathbf{b}_1, \mathbf{b}_3 \rangle = 0 \) and \( \langle \mathbf{b}_2, \mathbf{b}_3 \rangle = 0 \) in a similar way; I leave the details to the reader.

(b) (10 points) Write the vector \((3, -2, 1)\) in terms of the above vectors.

**Solution:**

Since the \( \mathbf{b}_i \) are orthonormal, if \( (3, -2, 1) = \sum c_i \mathbf{b}_i \), we have \( c_i = \langle (3, -2, 1), \mathbf{b}_i \rangle \). In other words
\[
c_1 = (3, -2, 1) \cdot (1/3, 2/3, 2/3) = 1/3
\]
\[
c_2 = (3, -2, 1) \cdot (-2/3, -1/3, 2/3) = -2/3
\]
\[
c_3 = (3, -2, 1) \cdot (2/3, -2/3, 1/3) = 11/3.
\]

Therefore \( (3, -2, 1) = \frac{1}{3} \mathbf{b}_1 - \frac{2}{3} \mathbf{b}_2 + \frac{11}{3} \mathbf{b}_3 \).

7. (15 points) In \( L^2[0, 1] \), apply the Gram-Schmidt process to the list of vectors \( 1, x \).

**Solution:**

We have \( w_1 = 1 \). For \( w_2 \), we get
\[ w_2 = x - \text{proj}_1 x. \]

Recall that \( \text{proj}_1 x = \frac{\langle 1, x \rangle}{\langle 1, 1 \rangle} \cdot 1 \). We have
\[
\langle 1, x \rangle = \int_0^1 1 \cdot x \, dx = \frac{x^2}{2} \bigg|_0^1 = \frac{1}{2}
\]
and
\[
\langle 1, 1 \rangle = \int_0^1 1 \cdot 1 \, dx = x \bigg|_0^1 = 1.
\]

Therefore \( \text{proj}_1 x = \frac{1/2}{1} 1 = \frac{1}{2} \). It follows that
\[ w_2 = x \left(1 - \frac{1}{2}\right) = x - \frac{1}{2}. \]