1. In \( \mathbb{R}^3 \), let \( \mathbf{u} = (2, -1, 1) \) and \( \mathbf{w} = (3, 2, -1) \).

Compute each of the following.

(a) (5 points) \(2\mathbf{u} - \mathbf{w}\)

Solution:

\[
(2 \cdot 2 - 3, 2 \cdot (-1) - 2, 2 \cdot 1 - (-1)) = (1, -4, 3).
\]

(b) (5 points) \( \langle \mathbf{u}, \mathbf{w} \rangle \)

Solution:

\[
2 \cdot 3 + (-1) \cdot 2 + 1 \cdot (-1) = 3.
\]

(c) (5 points) \( \| \mathbf{u} \|^2 \)

Solution:

\[
\| \mathbf{u} \|^2 = \langle \mathbf{u}, \mathbf{u} \rangle = 2^2 + (-1)^2 + 1^2 = 6.
\]

2. (a) (10 points) Find a Cartesian equation for the plane through the point \((3, 4, -1)\) which is perpendicular to the line with equation \( \mathbf{r} = (0, 1, -1) + t(-2, 2, 1) \).

Solution: The direction vector for the line, \((-2, 2, 1)\), must be orthogonal to the plane, so we can use it as a normal vector \( \mathbf{n} \). We are given the point \((3, 4, -1)\), so we set it equal to \( \mathbf{r}_0 \). Then we get \( \mathbf{n} \cdot \mathbf{r}_0 = -6 + 8 - 1 = 1 \), so our equation is \((-2, 2, 1) \cdot \mathbf{r} = 1 \), or alternatively

\[-2x + 2y + z = 1.\]

(b) (15 points) Find a Cartesian equation for the plane which contains the lines \( \mathbf{r} = (1, 2, 1) + t(-1, 1, 0) \) and \( \mathbf{r} = (4, -1, 1) + t(3, 1, 2) \).
Solution: Any normal vector \( n \) for the plane must be perpendicular the direction vectors for the lines; thus \( n \cdot (-1, 1, 0) = 0 \) and \( n \cdot (3, 1, 2) = 0 \). We set up the system of equations for \( n \); the associated matrix is

\[
\begin{bmatrix}
-1 & 1 & 0 \\
3 & 1 & 2
\end{bmatrix}
\]

(I omit the augmentation, since it is all zero.) We put this in ref and obtain the solution \( n = (1, 1, -2) \) (work omitted, but you can easily check the answer). Our \( r_0 \) could be \((1, 2, 1)\), and we have \( n \cdot r_0 = 1+2-2 = 1 \). Of course, if we pick \( r_0 = (4, -1, 1) \), we get the same dot product; otherwise, we’d have a problem! In any case, our equation is \((1, 1, -2) \cdot r = 1\), or \(x + y - 2z = 1\).

3. (a) (15 points) Find the area of the triangle with vertices at \((2, -1), (1, 4), \text{ and } (5, 2)\).

Solution: Label our points, in order, \(P, Q, R\). Then the triangle is defined by the 2 vectors \(\overrightarrow{PQ} = (1, 4) - (2, -1) = (-1, 5)\) and \(\overrightarrow{PR} = (5, 2) - (2, -1) = (3, 3)\). Our triangle has half the area of the parallelogram defined by these two vectors, and hence it is

\[
\frac{1}{2} \left| \begin{array}{cc}
-1 & 5 \\
3 & 3
\end{array} \right| = \frac{1}{2} |-3 - 15| = 9.
\]

(b) (5 points) Compute \(\det\left[ \begin{array}{cc}
-2 & 5 \\
3 & 2
\end{array} \right] \).

Solution: It is \(-4 - 15 = -19\).

(c) (10 points) Given a \(3 \times 3\) matrix \(A\), a new matrix \(B\) is obtained from \(A\) by multiplying all of the entries of \(A\) by 3. Write an equation relating \(\det B\) to \(\det A\), and justify that the equation is correct.

Solution: We have \(\det B = 27 \det A\). The reason is that multiplying \(A\) by 3 is the same as multiplying each row of \(A\) by 3. Each row multiplication also multiplies the determinant by 3. Since there are 3 rows, the total multiplication of the determinant is \(3^3 = 27\).

4. (a) (15 points) Determine if the list

\[(1, 1, -3), (2, -3, 4), (7, -3, -1)\]

is linearly dependent or linearly independent. If it is dependent, eliminate the fewest number of vectors possible to obtain a linearly independent set. Make sure to justify that the resulting set is linearly independent!

Solution: We set up the matrix

\[
\begin{bmatrix}
1 & 2 & 7 \\
1 & -3 & -3 \\
-3 & 4 & -1
\end{bmatrix}
\]
and row reduce. It turns out that the rref is
\[
\begin{bmatrix}
1 & 0 & 3 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}
\]
Thus the system is linearly dependent. The 3rd column is free, so we can omit it. The remaining pair \((1, 1, -3), (2, -3, 4)\) must form a linearly independent set.

(b) (5 points) Compute
\[
\det \begin{bmatrix}
1 & 2 & 7 \\
1 & -3 & -3 \\
-3 & 4 & -1
\end{bmatrix}.
\]
**Solution:** From the previous problem, this matrix has rank 2, and hence its determinant is 0.

(c) (10 points) Determine if the list of functions \(e^{ix}, \cos x, \sin x\) is linearly dependent or independent. Justify your answer.

**Solution:** One can compute a Wronskian, but it is easier to observe that
\[e^{ix} = \cos x + i \sin x\]
which shows the list is linearly dependent.

5. Consider the vectors
\[b_1 = (1/3, 2/3, 2/3) \quad b_2 = (-2/3, -1/3, 2/3) \quad b_3 = (2/3, -2/3, 1/3).
\]
(a) (20 points) Show that \(b_1, b_2, b_3\) forms an orthonormal list.

**Solution:** We have
\[
\|b_1\|^2 = \frac{1}{3^2} + \frac{2^2}{3^2} + \frac{2^2}{3^2} = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1.
\]
Therefore \(b_1\) is a unit vector. Similar calculations hold for \(b_2\) and \(b_3\). Furthermore,
\[
\langle b_1, b_2 \rangle = \frac{1 \cdot (-2)}{9} + \frac{2 \cdot (-1)}{9} + \frac{2 \cdot 2}{9} = \frac{-2 - 2 + 4}{9} = 0.
\]
Therefore \(b_1\) and \(b_2\) are orthogonal. One should check \(\langle b_1, b_3 \rangle = 0\) and \(\langle b_2, b_3 \rangle = 0\) in a similar way; I leave the details to the reader.

(b) (10 points) Write the vector \((1, \frac{1}{2}, -3)\) in terms of the vectors \(b_1, b_2, b_3\).
Solution: Label the last 3 vectors $v_1, v_2, \text{and }v_3$, respectively. The $v_i$ are identical to those given in one of the homework problems; as in that problem, the three vectors form an orthonormal basis for $\mathbb{R}^3$. Therefore if $(1, \frac{1}{2}, -3) = \sum c_i v_i$, we have $c_i = (1, \frac{1}{2}, -3) \cdot v_i$. In other words

$$ c_1 = (1, 1/2, -3) \cdot (1/3, 2/3, 2/3) = -4/3 $$
$$ c_2 = (1, 1/2, -3) \cdot (-2/3, -1/3, 2/3) = -17/6 $$
$$ c_3 = (1, 1/2, -3) \cdot (2/3, -2/3, 1/3) = -2/3. $$

Therefore

$$ \left(1, \frac{1}{2}, -3\right) = -\frac{4}{3} \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) - \frac{17}{6} \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) - \frac{2}{3} \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right). $$

6. (15 points) In $L^2[0,1]$, apply the Gram-Schmidt process to the list of vectors $1, x$.

Solution: Let $v_1 = 1, v_2 = x$. Then $w_1 = 1$, while

$$ w_2 = x - \text{proj}_1 x = x - \frac{(1, x)}{(1, 1)} \cdot 1. $$

Now we compute the inner products. We have

$$ (1, x) = \int_0^1 1 \cdot x \; dx $$
$$ = \frac{1}{2} x^2 \bigg|_0^1 $$
$$ = \frac{1}{2}. $$

while

$$ (1, 1) = \int_0^1 1 \cdot 1 \; dx $$
$$ = x \bigg|_0^1 $$
$$ = 1. $$

Therefore $w_2 = x - \frac{1/2}{1} \cdot 1 = x - \frac{1}{2}$. 