1. Let \( z = \frac{\sqrt{3}}{2} + \frac{1}{4}i \).

(a) (5 points) Compute \( (z + \overline{z})/2 \).

Solution: This is just the formula for \( \text{Re}(z) \), which in this case is \( \frac{\sqrt{3}}{2} \).

(b) (10 points) Write \( z \) in exponential form.

Solution: Graphing \( z \), we see that \( r = 1 \) and \( \theta = \pi/6 \). Therefore \( z = e^{\pi i/6} \).

(c) (10 points) Compute \( z^{90} \) in rectangular form.

Solution: We have
\[
  z^{90} = e^{90\pi i/6} = e^{15\pi i} = e^{7(2\pi i) + \pi i} = e^{\pi i} = -1.
\]

2. Compute the following. Unless specified, you may use the form of your choice.

(a) (10 points) \( (2 - i)^2 \)
Solution:

\[(2 - i)(2 - i) = 4 + i^2 - 2i - 2i = 3 - 4i.\]

(b) (10 points) \[\frac{e^{2\pi i/3}}{1 + i}\]

Solution: Division is easiest in exponential form. By graphing, we see that \[1 + i = \sqrt{2}e^{i\pi/4},\]
and so

\[
\frac{e^{2\pi i/3}}{1 + i} = \frac{e^{2\pi i/3}}{\sqrt{2}e^{i\pi/4}} = \frac{1}{\sqrt{2}}e^{\frac{2\pi i}{3} - \frac{i\pi}{4}} = \frac{1}{\sqrt{2}}e^{\frac{5\pi i}{12}}.
\]

(c) (10 points) \[\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) \cdot (-i)\] in exponential form

Solution: One can either multiply, and then convert to exponential, or vice versa. I think it’s easier to convert first. By graphing, we observe that

\[
\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = e^{i\pi/4}
\]
and \[-i = e^{3\pi i/2}.\] Multiplying the two, we get \[e^{i\left(\frac{3\pi}{2} + \frac{\pi}{4}\right)} = e^{\frac{7\pi i}{4}}.\]

(d) (10 points) \[e^{\pi i} + e^{3\pi i} + e^{5\pi i} + \ldots + e^{51\pi i}\]

Solution: If \(n\) is a whole number, then \(e^{(2n+1)\pi i} = (e^{2\pi i})^n \cdot (e^{\pi i}) = -1,\) so the above is just \(-1 - 1 - \ldots - 1 = -26.\)

3. Find all solutions for the following.

(a) (15 points) \(z^4 + 4 = 0\)

Solution: We have \(z^4 = -4.\) We know that there are 4 roots. Letting \(z = re^{i\theta},\) and observing that \(-4 = 4e^{i\pi} = 4e^{i3\pi} = 4e^{i5\pi} = 4e^{i7\pi},\) we see that \(r^4 = 4\) and \(4\theta\) is one of \(\pi, 3\pi, 5\pi, 7\pi.\) Thus \(r = \sqrt{2}\) and \(\theta\) is one of \(\pi/4, 3\pi/4, 5\pi/4, 7\pi/4.\) Our solutions are therefore

\[
\sqrt{2}e^{i\pi/4}, \sqrt{2}e^{i3\pi/4}, \sqrt{2}e^{i5\pi/4}, \sqrt{2}e^{i7\pi/4}.
\]

(b) (15 points) \(z^3 = 2i - 2\)
Solution: As above, let \( z = re^{i\theta} \). Observe that
\[
2i - 2 = 2\sqrt{2}e^{\frac{3\pi}{4}} = 2\sqrt{2}e^{\frac{11\pi}{4}} = 2\sqrt{2}e^{\frac{19\pi}{4}}.
\]
Thus \( r^3 = 2\sqrt{2} = 2^{3/2} \), and so \( r = \sqrt{2} \). Additionally, \( 3\theta = 3\pi/4 \) or \( 11\pi/4 \) or \( 19\pi/4 \), and so \( \theta = \pi/4 \) or \( 11\pi/12 \) or \( 19\pi/12 \). Therefore our solutions are
\[
\sqrt{2}e^{\frac{i\pi}{4}}, \sqrt{2}e^{\frac{i11\pi}{12}}, \sqrt{2}e^{\frac{i19\pi}{12}}.
\]

4. (15 points) Come up with a formula for \( \text{Im}(z) \) in terms of \( z \) and \( \overline{z} \), and prove that it is true.

Solution: We have \( \text{Im}(z) = \frac{1}{2i}(z - \overline{z}) \). If \( z = x + iy \), then \( \overline{z} = x - iy \), and we get
\[
\frac{1}{2i}(z - \overline{z}) = \frac{1}{2i}(x + iy - x + iy) = \frac{1}{2i}(2iy) = y = \text{Im}(z).
\]
This proves the identity.

5. Graph and describe with words the sets described by the following equations. Explain your answers.

(a) (10 points) \( \frac{z - \overline{z}}{2i} = -2 \).

Solution: This says \( \text{Im}(z) = -2 \), or in other words the line \( y = -2 \).

(b) (10 points) \( |z + 1 - i| = 4 \).

Solution: This is the set of points which is 4 units from \(-1 + i\), and therefore is a circle with radius 4 centered at \(-1 + i\).