Fourier Analysis
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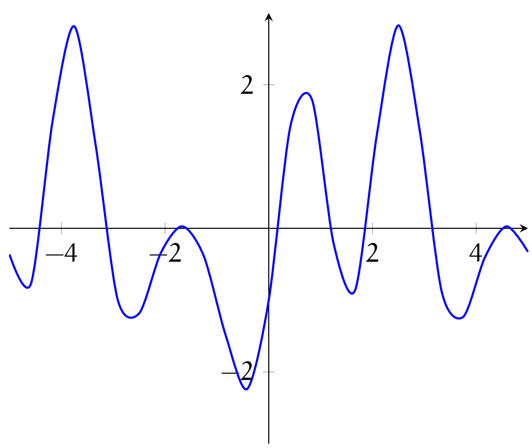
7 Fourier Analysis

7.1 Introduction

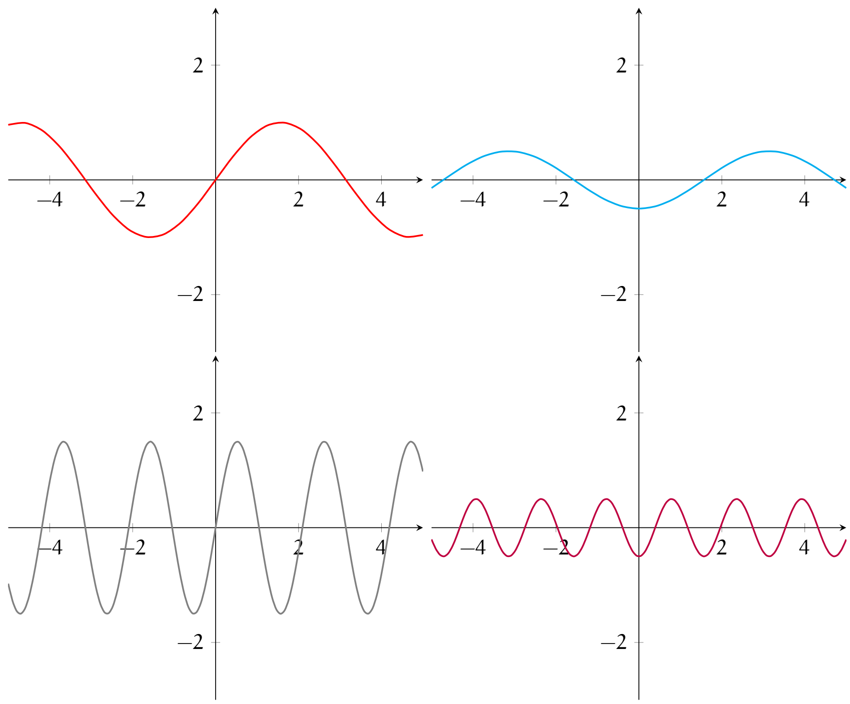
The purpose of this section is to decompose a complex signal into its constituent frequencies. To make this more concrete, consider the following examples:

- A physicist places a prism in a light beam, and the beam splits into the rainbow of colors.
- An astronomer observes the light from a distant star. They analyze the specific "colors" in the light (many of which are not visible to the eye) and deduces the chemical composition of the star.
- An engineer sets up an oscilloscope that receives a complex signal. The oscilloscope computes the frequency graph of the signal.

There are many others! The idea is the same in all cases. First, define a *simple signal* to be anything that looks like a sine wave. Then given a more complicated signal, we can rewrite it as a sum of simple signals. For instance, the signal given by the graph



looks complicated. But it turns out that it is the sum of the graphs given below:



Fourier analysis gives us a method for accomplishing the decomposition above.

The way our text explains Fourier analysis is uncharacteristically terrible. This document aims to give the *correct* way of thinking about Fourier analysis. It is by no means original to me; most textbooks handle Fourier analysis the way I will describe.

7.2 Preliminaries

We use a linear algebraic framework. First, we want to work with functions that repeat every 2π, such as $\sin x$, $\cos x$, $\sin 2x$, $\cos 2x$, etc.

Proposition 7.1. *A function f is 2π-periodic exactly when $f(x) = f(x + 2\pi)$.*

Notice that the equation means $f(0) = f(0 + 2\pi) = f(2\pi) = f(4\pi) = \cdots$, and similarly for any other value we plug in. Alternatively, the graph of $y = f(x + 2\pi)$ is the graph of $y = f(x)$ translated 2π units to the left. If $f(x)$ is 2π-periodic, the translation does not change the graph; hence $f(x) = f(x + 2\pi)$.

When working with a 2π-periodic function, the values of the function on $[-\pi, \pi]$ completely determine the function everywhere else. For instance, if f is 2π-periodic, then $f(\frac{11\pi}{2}) = f(\frac{3\pi}{2})$. Therefore instead of 2π-periodic functions with domain all real numbers, we can look at functions with domain $[-\pi, \pi]$.

We will silently identify 2π-periodic functions with functions having domain $[-\pi, \pi]$.

Next, recall that $L^2[-\pi,\pi]$ means the vector space of functions with domain the interval $[-\pi,\pi]$, with inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f}g\,dx.$$

Our main tool is the projection formula

$$\mathrm{proj}_f(g) = \frac{\langle f, g\rangle}{\langle f, f\rangle}f.$$

So for example if $f = x$ and $g = 1 + 2x$, then to compute $\mathrm{proj}_x(1 + 2x)$, we do

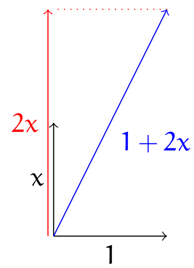
$$\begin{aligned}
\langle x, 1+2x\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{x}(1+2x)\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} x + 2x^2\,dx \text{ since } x \text{ is real on } [-\pi,\pi] \\
&= \frac{1}{2\pi}\left[\frac{1}{2}x^2 + \frac{2}{3}x^3\right]_{-\pi}^{\pi} \\
&= \frac{2\pi^2}{3}, \\
\langle x, x\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{x}x\,dx \\
&= \frac{1}{2\pi}\left[\frac{1}{3}x^3\right]_{-\pi}^{\pi} \\
&= \frac{\pi^2}{3}, \\
\mathrm{proj}_x(1+2x) &= \frac{\langle x, 1+2x\rangle}{\langle x, x\rangle}x \\
&= \frac{2\pi^2/3}{\pi^2/3}x \\
&= 2x.
\end{aligned}$$

This is actually quite a natural answer! The reasons is that 1 and x are orthogonal in $L^2[-\pi,\pi]$:

$$\langle 1, x\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{4\pi}x^2\bigg]_{-\pi}^{\pi} = 0.$$

Thus thinking of $1 + 2x = 1 \cdot 1 + 2 \cdot x$, we expect the component of $1 + 2x$ along x to be $2x$. If we were to picture this, it would looks something like this:[1]

[1]Note that if we did the similar $\mathrm{proj}_{x^2}(1 + 2x^2)$, we would *not* get $2x^2$. This is because 1 and x^2 are not orthogonal. Try it!



More precisely, we recall the relationship between orthogonal bases and the projection formula.

Theorem 7.2. *Suppose $\vec{v}_1, \ldots, \vec{v}_n$ forms an orthogonal basis for a vector space V. If $\vec{v} \in V$, then*
$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$
where
$$c_i\vec{v}_i = \text{proj}_{\vec{v}_i}\vec{v}.$$

In particular, note that
$$c_i = \frac{\langle \vec{v}_i, \vec{v} \rangle}{\langle \vec{v}_i, \vec{v}_i \rangle}.$$

Corollary 7.3. *Suppose $\vec{v}_1, \ldots, \vec{v}_n$ forms an orthonormal basis for a vector space V. If $\vec{v} \in V$, then*
$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$
where
$$c_i = \langle \vec{v}_i, \vec{w} \rangle.$$

Orthonormal means that in additional to orthogonal, $\langle \vec{v}_i, \vec{v}_i \rangle = \|\vec{v}_i\|^2 = 1$. Substituting into the theorem, we get the corollary.

7.3 Trigonometric Fourier series via projections

Theorem 7.4. *In $L^2[-\pi, \pi]$, the list of functions*
$$1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots$$
forms an orthogonal basis. Additionally,
$$\|1\|^2 = 1, \quad \|\cos mx\|^2 = \|\sin nx\|^2 = \frac{1}{2}.$$

The fact that the list is a *basis* means that given $f \in L^2[-\pi, \pi]$, we can write a *Fourier series* for f,
$$F(f) = a_0 \cdot 1 + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$$

for some scalars a_n, b_m. For the most part, we'll have $F(f) = f$ (with notable exceptions to be discussed later).

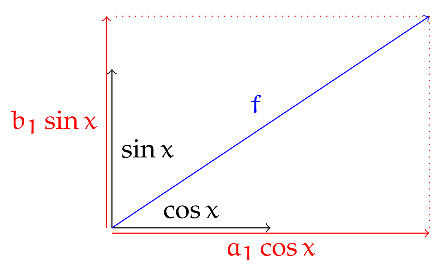
The fact that the list is *orthogonal* means that we can use Theorem 7.2 to compute the a_n and b_m using the projection formulas; namely

$$\begin{aligned} a_0 &= \frac{\langle 1, f\rangle}{\langle 1, 1\rangle} = \langle 1, f\rangle \\ a_n &= \frac{\langle \cos nx, f\rangle}{\langle \cos nx, \cos nx\rangle} = \frac{\langle \cos nx, f\rangle}{1/2} \\ b_m &= \frac{\langle \sin mx, f\rangle}{\langle \sin mx, \sin mx\rangle} = \frac{\langle \sin mx, f\rangle}{1/2}. \end{aligned}$$

Translating these into integrals, we get

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 \cdot f\, dx \\ a_n &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f \cos n\, dx\right) / (1/2) \\ b_m &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f \sin m\, dx\right) / (1/2) \end{aligned}$$

Here's a picture that's meant to illustrate this situation:



In the picture, the arrows representing $\cos x$ and $\sin x$ are perpendicular, which represents the fact that in $L^2[-\pi, \pi]$, $\cos x$ and $\sin x$ are orthogonal. The red vectors stand in for the projections of f onto the trig functions:

$$a_1 \cos x = \operatorname{proj}_{\cos x} f \quad \text{and} \quad b_1 \sin x = \operatorname{proj}_{\sin x} f.$$

The two red vectors sum to give the blue vector.

Note that the black vectors are *not* unit vectors. Instead, they have length $\frac{1}{\sqrt{2}}$.

This is a 2-dimensional picture. What we *actually* want is an infinite dimensional picture, with a black vector for each of $1, \cos x, \cos 2x, \ldots, \sin x, \sin 2x, \ldots$, each of them orthogonal to the others. But of course we can't actually draw that. However, the mathematics handles it just fine.

We call $F(f)$ the *trigonometric Fourier series* of f. We sometimes write $F_{\text{trig}}(f)$ for this Fourier series, since in the next section we will introduce an *exponential* Fourier series.

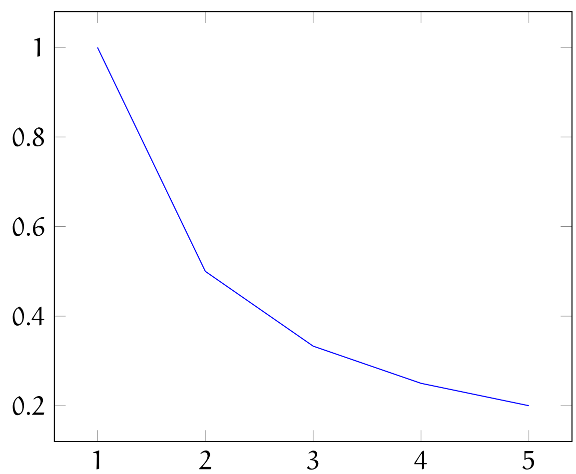
Finally, in summation notation we have

$$F(f) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{m=1}^{\infty} b_m \sin mx. \tag{1}$$

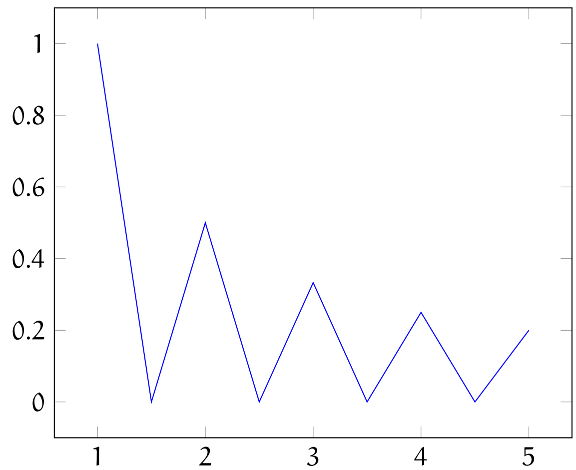
Frequency graphs. Especially in electrical engineering, the Fourier series is often represented graphically. This graph has x-axis the frequency (the "frequency domain") and y-axis the amplitude. Typically only the sine is considered. So in equation (1), we'd have a graph of the b_m values. For instance, if

$$F(f) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \cdots,$$

then we'd have a graph of the points $(1, 1)$, $(2, \frac{1}{2})$, $(3, \frac{1}{3})$, etc., which would look like this:



or maybe like this:



The difference, as you can see, is how the graph gets filled in between the m values ($m = 1, 2, 3, \ldots$). In practice, oscilloscopes use the *Fourier transform*, which we will cover at the end of the chapter. The Fourier transform allows us to deal with fractional multiples of the fundamental frequency.

7.4 Exponential Fourier series via projections

Theorem 7.5. *In $L^2[-\pi, \pi]$, the list of functions*

$$\ldots, e^{-2ix}, e^{-ix}, 1, e^{ix}, e^{2ix}, \ldots$$

forms an orthonormal basis.

Using the same reasoning as in the previous section, we see that any function $f \in L^2[-\pi, \pi]$ has *exponential Fourier series* given by

$$F(f) = \cdots + c_{-2}e^{-2ix} + c_{-1}e^{-ix} + c_0 \cdot 1 + c_1 e^{ix} + c_2 e^{2ix} + \cdots .$$

The fact that the e^{ikx} are orthonormal means that

$$c_k = \langle e^{ikx}, f \rangle .$$

Thus

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{e^{ikx}} f \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f \, dx.$$

Note that the conjugate on the first function actually matters with exponential Fourier series.

Our final series expansion could be written

$$F(f) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}.$$

7.5 Why do we do it this way?

Why this approach as opposed to the book's? For many reasons:

- The linear algebra picture is really what's going on!
- We can use properties of projections, which are not as easily available with Boas' methods. We talk about this in class in connection with geometric transformations of functions.
- It's easier to remember. Of course, you have to actually *understand* what's happening geometrically, which you should do, but then you have it.
- The approach generalizes. As mentioned on p. 357 of the textbook, the material in chapters 12 and 13 also follow the same framework.