Name:

Math 330: Exam 2

November 6, 2023

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Avoid using the back of the page; instead, there is an extra sheet at the end that you can use. Calculators are not allowed.

1. (a) (5 points) On a perfectly balanced seesaw are 3 kids. The first kid weighs 30 pounds and is on the left edge of the seesaw. The second kid weighs 40 pounds and is 3 feet from the left edge. The last kid weighs 50 pounds and is 8 feet from the left edge. How far from the left edge is the fulcrum?

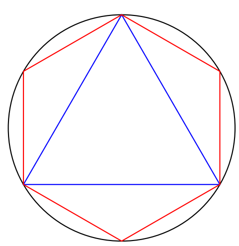
(a) ____________

Solution: It is the weighted average of the positions of the kids:

$$\frac{0 \cdot 30 + 3 \cdot 40 + 8 \cdot 50}{30 + 40 + 50} = \frac{520}{120} = \frac{13}{3}.$$

So $\frac{13}{3}$ feet.

(b) (10 points) Using one of Archimedes' methods, suppose you want to estimate the area of the circle shown starting with an equilateral triangle. Draw in the appropriate triangle, then draw in the shape that you'd use for the *next* approximation.



Solution: See above: I used an inscribed triangle and an inscribed hexagon. You could also use circumscribed shapes.

(c) (10 points) In the Archimedes crown problem, a crown is balanced on a scale with an equal mass of gold. The whole apparatus is placed underwater. What happens to the scale, and why?

Solution: If the crown is pure gold, nothing happens. If the crown is part silver, that pan goes up. The reason is the volume of the crown is greater than that of the pure gold, so the buoyant force of water is greater on that pan, pushing it up.

2. (10 points) Consider the parabola with equation $x = 2y^2$, and let $P = (2, 1)$. Find Q on the x-axis for which the line PQ is tangent to the parabola at P.

Solution: I did not forbid you from using calculus, so that is a totally valid method. But using Apollonius' method, the axis of symmetry is the x-axis. Projecting P onto this axis gives the point $(2, 0)$. The point Q is on the other side of and equidistant from the vertex $(0, 0)$; in other words, $Q = (-2, 0)$.

3. (10 points) A triangle has sides of length 4, 13, and 15. Find its area.

Solution: We use Heron's Formula. We have $s = \frac{1}{2}(4 + 13 + 15) = 16$, and so the area is

$$\begin{aligned}\sqrt{16 \cdot (16-4) \cdot (16-13) \cdot (16-15)} &= \sqrt{16 \cdot 12 \cdot 3 \cdot 1} \\ &= \sqrt{4^2 \cdot 2^2 \cdot 3^2} \\ &= 4 \cdot 2 \cdot 3 = 24.\end{aligned}$$

4. (10 points) Find a positive rational number x such that both $x+2$ and $x+6$ are perfect squares.

Solution: Let $u^2 = x + 6$ and $v^2 = x + 2$. Then

$$(u + v)(u - v) = u^2 - v^2 = (x + 6) - (x + 2) = 4.$$

Let us set $u + v = 4$ and $u - v = 1$. Adding the equations, we get $2u = 5$ or $u = \frac{5}{2}$. We have $x + 6 = (\frac{5}{2})^2$, so $x = \frac{25}{4} - 6 = \frac{1}{4}$.

5. (5 points) In what city did Hypatia live?

5. ______________

Solution: Alexandria

6. (10 points) Find 2 different positive values of x for which $x \equiv 4 \pmod 5$ and $x \equiv 2 \pmod{17}$.

Solution: Start with the smallest solution to the first congruence: $x = 4$. Adding 5 maintains that this solves the first congruence: $9, 14, 19, \ldots$ The first of these that solves the second congruence is 19. Thus 19 is our first solution. Adding a multiple of 5 preserves the property of solving the first congruence; adding a multiple of 17 does the same for the second congruence. So we can add $5 \cdot 17 = 85$ to still solve both, yielding $19 + 85 = 104$. Thus we get 19 and 104.

7. (10 points) Compute the cube root of 32768 using Aryabhata's method.

Solution: The work is hard to type out, but the answer is 32.

8. (a) (10 points) Given that $(2, 5)$ is a solution to $6x^2 + 1 = y^2$, find another solution.

Solution: According to Brahmagupta's Theorem, a new solution is given by

$$(2 \cdot 5 + 2 \cdot 5, 6 \cdot 2 \cdot 2 + 5 \cdot 5) = (20, 49).$$

(b) (5 points) What is the name of the Indian mathematician who discovered the method you used in part (a)?

(b) ____________

Solution: Brahmagupta.