Name:

Math 270: Final Exam

May 17, 2021

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Calculators are not allowed.

1. Negate each of the following.
   (a) (5 points) $\forall x \in \mathbb{Z}$, if x is composite, then x is a perfect square.
   (b) (5 points) $\exists y \in \mathbb{R}$ such that y is irrational and y^2 is rational.
   (c) (5 points) $\exists A, B \subseteq \mathbb{R}$ such that $\forall C \subseteq A$, $C \cap B = \varnothing$.
   (d) (5 points) $\forall a \in \mathbb{Z}$, $\exists b \in \mathbb{Z}$ such that $a - b^2 > 0$.
2. (20 points) Prove that given three consecutive integers, at least one of them must be divisible by 3.
3. (15 points) Show that the square root of any irrational number is irrational. (Hint: use indirect proof.)
4. (15 points) Show that for all $n \geq 1$,
$$\sum_{j=1}^{n}(2j-1) = n^2.$$
5. (15 points) Suppose that (g_n) is the sequence defined by
$$g_1 = 3,\ g_2 = 5 \text{ and } g_n = 3g_{n-1} - 2g_{n-2} \text{ for } n \geq 3.$$
   Prove that $g_n = 2^n + 1$ for all n.
6. (20 points) Given sets A, B, and C, prove that
$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$
7. Let S be the set of strings of 1s and 0s. (This includes the empty string.) Define
$$f : S \longrightarrow \mathbb{Z}^{nonneg}$$
   by $f(x) =$ the length of x as a string.
   (a) (10 points) Prove or disprove: f is one-to-one.
   (b) (10 points) Prove or disprove: f is onto.
8. For $x, y \in \mathbb{R} - \{0\}$, say $x \sim y$ if $xy > 0$.
   (a) (15 points) Prove that $\sim$ is an equivalence relation.
   (b) (5 points) Determine how many equivalence classes $\sim$ has, and list them. You do not have to prove your answer.