

Name:

Math 270: Exam 2

November 22, 2016

- Justify your reasoning in complete sentences.
- You may use any result proven in the sections covered up to this point, but you may not use homework problems.
- No notes or electronic aids are allowed.

Question	Points	Score
1	20	
2	15	
3	15	
4	15	
5	15	
6	20	
Total:	100	

1. Short answer questions. You don't have to justify any of your answers, and there is no extra credit.

(a) (5 points) True or false: The product of two irrational numbers is irrational.

(a) _____

Solution: False: $\sqrt{2} \cdot \sqrt{2} = 2$.

(b) (5 points) Compute $\prod_{j=1}^3 \frac{j+1}{2j}$.

(b) _____

Solution: This is $\frac{2}{2} \cdot \frac{3}{4} \cdot \frac{4}{6} = \frac{1}{2}$.

(c) (5 points) Let the universe U be \mathbb{Z} . If

$$A = \mathbb{Z}^+ \text{ and } B = \{n \in \mathbb{Z} \mid -3 \leq n \leq 10\},$$

how many elements are in $A^c \cap B$?

(c) _____

Solution: Four: $A^c \cap B = \{-3, -2, -1, 0\}$.

(d) (5 points) How many elements are in $\mathcal{P}(\mathcal{P}(\emptyset))$?

(d) _____

Solution: Two: $\mathcal{P}(\emptyset) = \{\emptyset\}$ so $\mathcal{P}(\mathcal{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$.

2. (15 points) Prove that for $n \geq 2$,

$$\sum_{k=1}^{n-1} k(k+1) = \frac{n(n-1)(n+1)}{3}.$$

Solution: We use induction. The base case is $n = 2$. In this case, the sum is $1(1+1) = 2$, while the fraction is $\frac{2(2-1)(2+1)}{3} = \frac{6}{3} = 2$. Thus the base case holds.

For the inductive step, suppose that

$$\sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3}$$

for some $k \geq 3$. Then

$$\begin{aligned} \sum_{i=1}^k i(i+1) &= \left(\sum_{i=1}^{k-1} i(i+1) \right) + k(k+1) \\ &= \frac{k(k-1)(k+1)}{3} + k(k+1) \end{aligned}$$

by the inductive hypothesis. The latter equals

$$\begin{aligned} \frac{k(k-1)(k+1)}{3} + \frac{3k(k+1)}{3} &= \frac{k(k-1)(k+1) + 3k(k+1)}{3} \\ &= \frac{k(k+1)(k-1+3)}{3} \\ &= \frac{k(k+1)(k+2)}{3}. \end{aligned}$$

But $(k+1)(k+1-1)(k+1+1) = k(k+1)(k+2)$, and so the inductive conclusion holds.

By induction, the identity is true.

3. (15 points) Prove that $\forall n \in \mathbb{Z}$ with $n \geq 2$, that n is divisible by at least one prime.

Solution: We use strong induction. If $n = 2$, then certainly $2 \mid 2$ and 2 is a prime.

Now suppose that for $k \geq 2$, if $2 \leq n \leq k$, then n is divisible by a prime. Consider $k+1$. If it is prime, then $(k+1) \mid (k+1)$, and the claim holds. If not, $k+1$ is composite, so $\exists a, b \in \mathbb{Z}^+$ such that $ab = k+1$ and neither a nor b is equal to 1 or $k+1$. In particular, this means that $2 \leq a \leq k$. By our inductive hypothesis, there is a prime p for which $p \mid a$. But $a \mid (k+1)$ by definition of divisibility, so by transitivity of divisibility $p \mid (k+1)$.

By strong induction, the statement is true.

4. (15 points) Find an explicit formula for the sequence defined by

$$a_0 = 3, \quad a_n = 7a_{n-1} + 3 \text{ for } n \geq 1.$$

Prove your answer.

Solution: By computing a_1, a_2, \dots , one observes that

$$\begin{aligned} a_n &= 7^n \cdot 3 + 7^{n-1} \cdot 3 + \dots + 7^1 \cdot 3 + 7^0 \cdot 3 \\ &= 3(7^n + 7^{n-1} + \dots + 7^0) \\ &= 3 \frac{7^{n+1} - 1}{7 - 1} \\ &= \frac{7^{n+1} - 1}{2}. \end{aligned}$$

We prove it by induction. The base case is $n = 0$. Note that $a_0 = 3$, but

$$\frac{7^{0+1} - 1}{2} = \frac{6}{2} = 3.$$

For the inductive step, suppose $a_k = \frac{7^{k+1}-1}{2}$. Then

$$\begin{aligned} a_{k+1} &= 7a_k + 3 \\ &= 7 \frac{7^{k+1} - 1}{2} + \frac{6}{2} \\ &= \frac{7^{k+2} - 7 + 6}{2} \\ &= \frac{7^{k+2} - 1}{2}. \end{aligned}$$

Here, the first equality is by definition of a_k , and the second is by inductive hypothesis.

By induction, the claim holds.

5. (15 points) Define a sequence c_n by

$$c_0 = 1, \quad c_1 = 2, \quad c_n = 2c_{n-1} + 3c_{n-2} \text{ for } n \geq 2.$$

Find an explicit formula for c_n .

Solution: The characteristic equation is

$$x^2 - 2x - 3 = 0.$$

This factors as $(x - 3)(x + 1) = 0$, so $x = 3$ or $x = -1$. Thus by our Theorem on 2nd order linear recurrences,

$$c_n = C \cdot 3^n + D \cdot (-1)^n.$$

Plugging in $n = 0$ and $n = 1$, we get

$$\begin{aligned}1 &= C + D \\2 &= 3C - D.\end{aligned}$$

Adding the two equations, we find $4C = 3$ so $C = \frac{3}{4}$. From the first equation, we have $D = \frac{1}{4}$. Therefore

$$c_n = \frac{3}{4}3^n + \frac{1}{4}(-1)^n = \frac{3^{n+1} + (-1)^n}{4}.$$

6. (20 points) Given sets A , B , and C , prove that

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

Solution: Let $(x, y) \in A \times (B \cap C)$. By definition of Cartesian product, $x \in A$ and $y \in B \cap C$. This means that $x \in A$, $y \in B$, and $y \in C$. Since $x \in A$ and $y \in B$, we have $(x, y) \in A \times B$. Similarly, $(x, y) \in A \times C$. Therefore $(x, y) \in (A \times B) \cap (A \times C)$. We conclude that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

Now let $(x, y) \in (A \times B) \cap (A \times C)$. This means that $(x, y) \in A \times B$, and $(x, y) \in A \times C$. The first statement means that $x \in A$ and $y \in B$, while the second means that $x \in A$ and $y \in C$. Therefore $x \in A$ and $y \in B \cap C$, so $(x, y) \in A \times (B \cap C)$. Hence $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.