Name:

Math 270: Exam 1

March 8, 2021

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Calculators are not allowed.

1. (15 points) Give the truth table for "If either P or Q, then R."

P	Q	R	$P \vee Q$	$(P\vee Q)\Rightarrow R$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	T
F	F	F	F	T

2. Negate the following. Write the negation in the simplest form possible (so for example, don't just stick a "not" out front!).

   (a) (5 points) The popcorn is both salty and yellow.

   Solution: The popcorn is either not salty or not yellow.

   (b) (5 points) If the creature has six legs, then it is an insect.

   Solution: The creature is six-legged but not an insect.

   (c) (5 points) There exist numbers x and y such that $x^3 + y^3$ is a perfect cube.

   Solution: For all numbers x, y, $x^3 + y^3$ is not a perfect cube.

   (d) (5 points) For all $n \in \mathbb{Z}$, $\exists m \in \mathbb{Q}$ such that $n + m$ is irrational.

   Solution: $\exists n \in \mathbb{Z}$ such that $\forall m \in \mathbb{Q}$, $n + m$ is rational.

   (e) (5 points) $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{Z}$, $x^2 + y \in \mathbb{Z}$.

   Solution: $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{Z}$ such that $x^2 + y \notin \mathbb{Z}$.

3. True or false, and prove your answer.

   (a) (10 points) $\exists x \in \mathbb{Z}$ such that $x^2 + 4$ is prime.

Solution: True: take $x = 1$. Then $1^2 + 4 = 5$ is prime.

(b) (10 points) $\forall t \in \mathbb{R}$, $t^2 \geq t$.

Solution: False: take $t = \frac{1}{2}$. Then $t^2 = \frac{1}{4}$, and $t^2 < t$.

4. (20 points) Prove that if x and y are rational numbers, then $2x + y$ is rational.

Solution: Suppose $x, y \in \mathbb{Q}$. By definition of rational, $\exists a, b, c, d \in \mathbb{Z}$, $b, d \neq 0$, such that $x = \frac{a}{b}$ and $y = \frac{c}{d}$. Then

$$\begin{aligned} 2x + y &= \frac{2a}{b} + \frac{c}{d} \\ &= \frac{2ad + bc}{bd}. \end{aligned}$$

Since sums and products of integers are integers, both $2ad + bc$ and bd are integers. By the Zero Product Property, $b, d \neq 0$ implies that $bd \neq 0$. By the definition of rational, $2x + y$ is rational.

5. (20 points) Prove that given three consecutive integers, at least one of them must be divisible by 3.

Solution: This solution was given in the practice problems: Let n be the smallest number, so the three numbers are $n, n+1$, and $n+2$. By the Quotient-Remainder Theorem, $\exists q, r \in \mathbb{Z}$ such that

$$n = 3q + r$$

with $0 \leq r \leq 2$; in other words, $r = 0, 1$, or 2. We will treat each of these as a different case.

Case 1: $r = 0$**.** In this case, $n = 3q$, so by definition of divisibility, $3 \mid n$, and so one of the numbers is divisible by 3.

Case 2: $r = 1$**.** In this case, $n = 3q + 1$, so $n + 2 = 3q + 3 = 3(q + 1)$. Since sums of integers are integers, $q + 1 \in \mathbb{Z}$. By definition of divisibility, $3 \mid (n + 2)$, and so the third number in our list is divisible by 3.

Case 3: $r = 2$**.** In this case, $n = 3q + 2$, so $n + 1 = 3q + 3 = 3(q + 1)$. The rest of the argument is basically the same as in case 2, so I leave the details to you.