Name:

Math 270: Exam 2

April 16, 2026

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Avoid using the back of the page; instead, there is an extra sheet at the end that you can use. Calculators are not allowed.

1. For the following short answer questions, you do not have to prove your answer.

   (a) (10 points) The sequence (a_n) is defined by $a_n = \frac{n}{n+1}$. Compute a_{100}.

   Solution:
   $$a_{100} = \frac{100}{101}$$

   (b) (10 points) The sequence (b_n) is defined by $b_0 = 0$ and $b_n = b_{n-1} + (-1)^n$ for $n \geq 1$. Compute b_{99}.

   Solution: We observe that
   $$\begin{aligned} b_0 &= 0 \\ b_1 &= 0 + (-1) = -1 \\ b_2 &= -1 + (+1) = 0 \\ b_3 &= 0 + (-1) = -1 \\ b_4 &= -1 + (+1) = 0. \end{aligned}$$
   The pattern is the even-numbered ones are 0, and the odd-numbered ones are -1. So $b_{99} = -1$.

   (c) (10 points) The sequence (c_n) is defined by $c_0 = 2$ and $c_{n+1} = 2 + \sqrt{c_n}$. Assume $\lim_{n\to\infty} c_n$ exists. Compute the limit.

   Solution: We have
   $$\lim c_{n+1} = 2 + \sqrt{\lim c_n}.$$
   Let $c = \lim c_n = \lim c_{n+1}$. Then
   $$c = 2 + \sqrt{c}.$$
   Thus $\sqrt{c} = c - 2$, so
   $$c = (c-2)^2, \text{ or } c = c^2 - 4c + 4.$$
   Therefore $c^2 - 5c + 4 = 0$, so $(c-4)(c-1) = 0$. Therefore $c = 1$ or $c = 4$. The recursion shows that $c_n \geq 2$ for all n, so the limit must be 4.

2. (15 points) Prove by induction that, for all $n \geq 0$,
$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}.$$

Solution: The base case is $n = 0$. The left hand side is just 1, while the right hand side is

$$2 - \frac{1}{2^0} = 2 - 1 = 1.$$

The two sides are equal, so the base case holds.

For the inductive step, suppose that

$$1 + \frac{1}{2} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$$

for some n. Adding $\frac{1}{2^{n+1}}$ to both sides, we get

$$\begin{aligned} 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} &= 2 - \frac{1}{2^n} + \frac{1}{2^{n+1}} \\ &= 2 - \frac{2}{2^{n+1}} + \frac{1}{2^{n+1}} \\ &= 2 + \frac{1-2}{2^{n+1}} \\ &= 2 - \frac{1}{2^{n+1}}. \end{aligned}$$

Therefore

$$1 + \frac{1}{2} + \cdots + \frac{1}{2^{n+1}} = 2 - \frac{1}{2^{n+1}},$$

and so by induction the identity holds.

3. (15 points) The sequence (d_n) is defined by $d_0 = 2$ and $d_{n+1} = d_n + 3$ for $n \geq 0$. Find a formula for d_n, and prove that your answer is correct.

Solution: Through some data collection, you should guess that $d_n = 2 + 3n$. Now we prove our answer with induction.

The base case is $n = 0$. Our formula gives $2 + 3 \cdot 0 = 2$, which does equal d_0.

For the inductive step, suppose that $d_n = 2 + 3n$ for some n. We know from definition of our sequence that $d_{n+1} = d_n + 3$. Substituting for d_n yields

$$\begin{aligned} d_{n+1} &= (2 + 3n) + 3 \\ &= 2 + (3n + 3) \\ &= 2 + 3(n + 1). \end{aligned}$$

Therefore the inductive conclusion holds, and so by induction our formula is correct.

4. (15 points) Prove that $7^n - 1$ is divisible by 6 for $n \geq 1$.

Solution: We use induction. The base case is $n = 1$. We have $7^1 - 1 = 6$, which is in fact divisible by 6.

Now suppose $7^n - 1$ is divisible by 6 for some n. By definition of divisible, $\exists k \in \mathbb{Z}$ such that

$7^n - 1 = 6k$. Then

$$\begin{aligned} 7^{n+1} - 1 &= 7 \cdot 7^n - 1 \\ &= 7 \cdot 7^n - 7^n + 7^n - 1 \\ &= (7-1) \cdot 7^n + (7^n - 1) \\ &= 6 \cdot 7^n + 6k \\ &= 6 \cdot (7^n + 6k). \end{aligned}$$

Since $k \in \mathbb{Z}$ and sums and products of integers are integers, $7^n + 6k$ is an integer. Therefore by definition of divisibility, $7^{n+1} - 1$ is divisible by 6.

By induction, our claim is true $\forall n \geq 1$.

5. (15 points) The sequence (f_n) is defined by

$$f_0 = 1, f_1 = 2, \text{ and } f_{n+2} = 2f_{n+1} + 3f_n \text{ for } n \geq 0.$$

Find an explicit formula for f_n.

Solution: We set up our quadratic equation

$$x^2 - 2x - 3 = 0.$$

The roots are $x = 3$ and $x = -1$. Therefore the general solution is

$$f_n = A \cdot 3^n + B \cdot (-1)^n.$$

Plugging in $n = 0$ and $n = 1$ yields

$$\begin{aligned} 1 &= A + B \text{ and} \\ 2 &= 3A - B. \end{aligned}$$

Adding the two equations gives $4A = 3$, so $A = \frac{3}{4}$. Plugging into the first equation, we get $B = \frac{1}{4}$. Therefore

$$f_n = \frac{3}{4} \cdot 3^n + \frac{1}{4} \cdot (-1)^n.$$