Name:

Math 270: Exam 1

March 5, 2026

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Avoid using the back of the page; instead, there is an extra sheet at the end that you can use. Calculators are not allowed.

1. (15 points) Give the truth table for "If P, then Q and R."

Solution:

P	Q	R	Q ∧ R	P ⇒ Q ∧ R
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

2. Negate the following. Write the negation in the simplest form possible (so for example, don't just stick a "not" out front!).

(a) (5 points) The book is either hardcover or flimsy.

Solution: The book is not hardcover and not flimsy.

(b) (5 points) If the mug has coffee in it, then the mug is hot.

Solution: The mug has coffee in it and is not hot.

(c) (5 points) $\forall n \in \mathbb{Z}$, $\sqrt{n^2+1}$ is irrational.

Solution: $\exists n \in \mathbb{Z}$ such that $\sqrt{n^2+1}$ is rational.

(d) (5 points) $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{Z}$, $x^2 + y > 0$.

Solution: $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{Z}$ such that $x^2 + y \leq 0$.

(e) (5 points) $\forall n \in \mathbb{Z}$, $\exists m \in \mathbb{Z}$ such that $n + 2m$ is prime.

Solution: $\exists n \in \mathbb{Z}$ such that $\forall m \in \mathbb{Z}$, $n + 2m$ is not prime.

3. True or false, and prove your answer.

(a) (10 points) $\forall m, n \in \mathbb{Z}$, $m^2 - n^2$ is even.

Solution: False: let $m = 1$ and $n = 0$. Then $m^2 - n^2 = 1^2 - 0^2 = 1$, which is not even.

(b) (10 points) $\exists n \in \mathbb{Z}$ such that both n and $n + 2$ are prime.

Solution: True: let $n = 3$. Observe that 3 and $3 + 2 = 5$ are both prime.

4. (20 points) Prove that if x and y are rational numbers, then their average is rational.

Solution: By definition of rational, $\exists a, b, c, d \in \mathbb{Z}$, $b, d \neq 0$, such that

$$x = \frac{a}{b} \text{ and } y = \frac{c}{d}.$$

The average of x and y is

$$\begin{aligned} \frac{x+y}{2} &= \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) \\ &= \frac{1}{2} \cdot \frac{ad+bc}{bd} \\ &= \frac{ad+bc}{2bd}. \end{aligned}$$

As sums and products of integers are integers, $ad + bc \in \mathbb{Z}$ and $2bd \in \mathbb{Z}$. Additionally, $2bd \neq 0$ by the Zero Product Property. Therefore by definition of rational, the average is rational.

5. (20 points) Prove that if $n \in \mathbb{Z}$ and n is not a multiple of 3, then $n^2 \mod 3 = 1$. (Hint: use cases.)

Solution: By the Quotient-Remainder Theorem, the possible remainders on division by 3 are $0, 1, 2$. Since n is not a multiple of 3, $n \mod 3 = 1$ or $n \mod 3 = 2$. We split into these two cases.

Case 1: $n \mod 3 = 1$. By the Quotient-Remainder Theorem, this means $\exists q \in \mathbb{Z}$ such that $n = 3q + 1$. Then

$$\begin{aligned} n^2 &= (3q+1)^2 \\ &= 9q^2 + 6q + 1 \\ &= 3(3q^2 + 2q) + 1. \end{aligned}$$

Sums and products of integers are integers, so $3q^2 + 2q \in \mathbb{Z}$. By the Quotient-Remainder Theorem, $n^2 \mod 3 = 1$.

Case 2: $n \mod 3 = 2$. By the Quotient-Remainder Theorem, this means $\exists q \in \mathbb{Z}$ such that $n = 3q + 2$. Then

$$\begin{aligned} n^2 &= (3q+2)^2 \\ &= 9q^2 + 12q + 4 \\ &= 3(3q^2 + 4q + 1) + 1. \end{aligned}$$

Sums and products of integers are integers, so $3q^2+4q+1 \in \mathbb{Z}$. By the Quotient-Remainder Theorem, $n^2 \mod 3 = 1$.