Solutions to Assignment 5

due: On gradescope 11:00PM, Thursday, April 10, 2025

1. Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be given by $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ -2y \\ -x+3y \end{bmatrix}$.

   Show that T is a linear transformation by verifying the linearity properties.

   Suppose first that $\vec{v}_1, \vec{v}_2 \in \mathbf{R}^2$, and that $\vec{v}_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$. We get that

   $$T(\vec{v}_1+\vec{v}_2) = T \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix} = \begin{bmatrix} (x_1+x_2)+(y_1+y_2) \\ -2(y_1+y_2) \\ -(x_1+x_2)+3(y_1+y_2) \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ -2y_1 \\ -x_1+3y_1 \end{bmatrix} + \begin{bmatrix} x_2+y_2 \\ -2y_2 \\ -x_2+3y_2 \end{bmatrix} = T(\vec{v}_1)+T(\vec{v}_2).$$

   Now suppose that $\vec{v} \in \mathbf{R}^2$, so $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, and that $c \in \mathbf{R}$ is a scalar. Then

   $$T(c\vec{v}) = T \begin{bmatrix} cx \\ cy \end{bmatrix} = \begin{bmatrix} cx+cy) \\ -2cy \\ -cx+3cy \end{bmatrix} = c \begin{bmatrix} x+y \\ -2y \\ -x+3y \end{bmatrix} = cT(\vec{v}),$$ as needed.

2. Show that T from Problem 1 is a matrix transformation by finding a matrix A such that $T(\vec{x}) = A\vec{x}$.

   Observe that $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ -2y \\ -x+3y \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, so that T is just the matrix transformation T_A for $A = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ -1 & 3 \end{bmatrix}$.

3. Give a counterexample to show that $T : \mathbf{R}^2 \to \mathbf{R}^2$ with $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-2 \\ y+1 \end{bmatrix}$ is not a linear transformation.

   There are many different ways for violating the linearity properties here. For example $2T(\vec{e}_1) = 2 \begin{bmatrix} 1-2 \\ 0+1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, whereas $T(2\vec{e}_1) = \begin{bmatrix} 2-2 \\ 0+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 2T(\vec{e}_1)$.

4. Let $\vec{d} \in \mathbf{R}^n$ be fixed, and consider the transformation $T_{\vec{d}} : \mathbf{R}^n \to \mathbf{R}$ given by $T_{\vec{d}}(\vec{v}) = \vec{d} \cdot \vec{v}$.

   (a) Show that $T_{\vec{d}}$ is a linear transformation by verifying the linearity properties.
   (b) Show that $T_{\vec{d}}$ is a matrix transformation, by finding a matrix A such that $T_{\vec{d}}(\vec{v}) = A\vec{v}$.

   (a) Let $\vec{v}_1, \vec{v}_2 \in \mathbf{R}^n$. Then $T_{\vec{d}}(\vec{v}_1+\vec{v}_2) = \vec{d} \cdot (\vec{v}_1+\vec{v}_2) = (\vec{d}\cdot\vec{v}_1) + (\vec{d}\cdot\vec{v}_2) = T_{\vec{d}}(\vec{v}_1) + T_{\vec{d}}(\vec{v}_2)$, as desired. Let $c \in \mathbf{R}$ and $\vec{v} \in \mathbf{R}^n$. Then $T_{\vec{d}}(c\vec{v}) = \vec{d} \cdot (c\vec{v}) = c(\vec{d}\cdot\vec{v}) = cT_{\vec{d}}(\vec{v})$, as desired.
   (b) Observe that $T_{\vec{d}}(\vec{v}) = \vec{d} \cdot \vec{v} = \vec{d}^T\vec{v}$, so that $T_{\vec{d}}$ is a matrix transformation with matrix $\vec{d}^T$.

5. Recall that $\text{proj}_{\vec{d}}(\vec{v}) = \frac{\vec{d}\cdot\vec{v}}{\vec{d}\cdot\vec{d}}\,\vec{d}$ is the projection of $\vec{v}$ onto $\vec{d}$.

   (a) Use Problem 4 to show that for fixed $\vec{d} \in \mathbf{R}^n$, $\text{proj}_{\vec{d}} : \mathbf{R}^n \to \mathbf{R}^n$ is a linear transformation.

   (b) Find the matrix for the transformation $\text{proj}_{\vec{d}} : \mathbf{R}^3 \to \mathbf{R}^3$ when $\vec{d} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

   (a) Let $\vec{v}_1, \vec{v}_2 \in \mathbf{R}^n$. Then using Problem 4, $\text{proj}_{\vec{d}}(\vec{v}_1+\vec{v}_2) = \frac{T_{\vec{d}}(\vec{v}_1+\vec{v}_2)}{\vec{d}\cdot\vec{d}}\vec{d} = \frac{T_{\vec{d}}(\vec{v}_1)+T_{\vec{d}}(\vec{v}_2)}{\vec{d}\cdot\vec{d}}\vec{d} = \left(\frac{T_{\vec{d}}(\vec{v}_1)}{\vec{d}\cdot\vec{d}} + \frac{T_{\vec{d}}(\vec{v}_2)}{\vec{d}\cdot\vec{d}}\right)\vec{d} = \frac{T_{\vec{d}}(\vec{v}_1)}{\vec{d}\cdot\vec{d}}\vec{d} + \frac{T_{\vec{d}}(\vec{v}_2)}{\vec{d}\cdot\vec{d}}\vec{d} = \text{proj}_{\vec{d}}(\vec{v}_1) + \text{proj}_{\vec{d}}(\vec{v}_2)$, as desired.
   Now let $\vec{v} \in \mathbf{R}^n$ and $\in \mathbf{R}$ be a scalar. Again using Problem 4,
   $\text{proj}_{\vec{d}}(c\vec{v}) = \frac{T_{\vec{d}}(c\vec{v})}{\vec{d}\cdot\vec{d}}\vec{d} = \frac{cT_{\vec{d}}(\vec{v})}{\vec{d}\cdot\vec{d}}\vec{d} = c\frac{T_{\vec{d}}(\vec{v})}{\vec{d}\cdot\vec{d}}\vec{d} = c\text{proj}_{\vec{d}}(\vec{v})$, as desired.

(b) For $\vec{d} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ we have $\operatorname{proj}_{\vec{d}}(\vec{e}_1) = \frac{\vec{d}\cdot\vec{e}_1}{\vec{d}\cdot\vec{d}}\vec{d} = \frac{2\cdot1-1\cdot0+3\cdot0}{2^2+(-1)^2+3^2}\vec{d} = \frac{1}{7}\vec{d}$. Similarly $\operatorname{proj}_{\vec{d}}(\vec{e}_2) = -\frac{1}{14}\vec{d}$, and $\operatorname{proj}_{\vec{d}}(\vec{e}_3) = \frac{3}{14}\vec{d}$, so that the standard matrix is

$$[\operatorname{proj}_{\vec{d}}] = [\operatorname{proj}_{\vec{d}}(\vec{e}_1)\ \operatorname{proj}_{\vec{d}}(\vec{e}_2)\ \operatorname{proj}_{\vec{d}}(\vec{e}_3)] = \frac{1}{14}\begin{bmatrix} 4 & -2 & 6 \\ -2 & 1 & -3 \\ 6 & -3 & 9 \end{bmatrix}.$$

6. Find the standard matrix for the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ that corresponds to a reflection about the line $y = -x$.

   Observe that reflecting $\vec{e}_1$ about $y = -x$ yields the vector $-\vec{e}_2$, whereas reflecting $\vec{e}_2$ yields $-\vec{e}_1$. Thus the standard matrix for T is $[T] = [T(\vec{e}_1)\ T(\vec{e}_2)] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

7. Find the standard matrix for the linear transformation that consists of a reflection about the y-axis, followed by a counterclockwise rotation by 30°, followed by a reflection about the line $y = x$

   With T be reflection across the y-axis, R_{30° the rotation by $\theta = 30^\circ$ counterclockwise, and S be the reflection about the line $y = x$, we see that our transformation is $S \circ R_{30^\circ} \circ T$. We have

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix},$$

   and thus its standard matrix is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Similarly, the standard matrix for S is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Thus the matrix for $S \circ R_{30^\circ} \circ T$ is

$$\begin{aligned}
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.
\end{aligned}$$

   (Note that this is the same as the matrix for rotation by 240°.)

8. In $\mathbf{R}^n$, let $\vec{d}$ be a nonzero vector. Let A be the matrix representing the linear transformation $\operatorname{proj}_{\vec{d}}$. Show that $A^2 = A$.

   First note that no matter what $\vec{v} \in \mathbf{R}^n$ we pick, $\operatorname{proj}_{\vec{d}}\vec{v} = c\vec{d}$ for some scalar c. Then

$$\begin{aligned}
\operatorname{proj}_{\vec{d}}(c\vec{d}) &= \frac{c\vec{d}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\vec{d} \\
&= c\vec{d}.
\end{aligned}$$

   This shows that $\operatorname{proj}_{\vec{d}} \circ \operatorname{proj}_{\vec{d}} = \operatorname{proj}_{\vec{d}}$. Translating this into standard matrices, we get $A^2 = A$.

9. Suppose $T : \mathbf{R}^4 \to \mathbf{R}^3$ is a linear transformation satisfying

$$T([1, -2, -1, 3]) = [2, -1, 1] \text{ and } T([-1, 1, 4, -2]) = [-1, 3, 4].$$

   Compute $T([1, -3, 2, 4])$.

   This problem is impossible to do *unless* $[1, -3, 2, 4]$ is in the span of $[1, -2, -1, 3]$ and $[-1, 1, 4, -2]$. Well, certainly the problem *is* possible, so we figure out the coefficients:

$$c_1\begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 4 \end{bmatrix}$$

translates to the system

$$\left[\begin{array}{rr|r} 1 & -1 & 1 \\ -2 & 1 & -3 \\ -1 & 4 & 2 \\ 3 & -2 & 4 \end{array}\right].$$

Taking the rref with MATLAB, we conclude that $c_1 = 2$ and $c_2 = 1$. Using the fact that T is linear, we get

$$\begin{aligned} T([1,-3,2,4]) &= T(2[1,-2,-1,3] + 1[-1,1,4,-2]) \\ &= 2T([1,-2,-1,3]) + T([-1,1,4,-2]) \\ &= 2[2,-1,1] + [-1,3,4] \\ &= [3,1,6]. \end{aligned}$$

10. In $\mathbf{R}^3$, let ℓ be the line given by $\vec{x} = t(1,1,1)$. Give the matrix which gives rotation by $120°$ about ℓ (in either direction).

    The rotation permutes the standard basis vectors; the specific direction determines which permutation. Let R the choice of rotation for which $R(\vec{e}_1) = \vec{e}_2$. Then $R(\vec{e}_2) = \vec{e}_3$ and $R(\vec{e}_3) = \vec{e}_1$. Therefore the matrix for R is

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$