Assignment 4

due: On gradescope 11:00PM, Thursday, March 20, 2025

1. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $\vec{b}_1 = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, and $\vec{b}_3 = \begin{bmatrix} -12 \\ 7 \end{bmatrix}$.

   (a) Find A^{-1} and use it to solve the three systems $A\vec{x} = \vec{b}_1$, $A\vec{x} = \vec{b}_2$, and $A\vec{x} = \vec{b}_3$.

   (b) Solve all three systems simultaneously by row reducing the augmented matrix $[A \mid \vec{b}_1\ \vec{b}_2\ \vec{b}_3]$ using Gauss-Jordan elimination.

2. Give a counterexample to show that in general $(A+B)^{-1} \neq A^{-1} + B^{-1}$.

3. (a) If A is a 3×5 matrix, explain why the columns of A must be linearly dependent.

   (b) If A is a 5×3 matrix, explain why the rows of A must be linearly dependent.

4. Let A, B be $n \times n$ matrices. We showed that in Theorem 3.9.d that if A and B are invertible, then so is AB. Prove the converse of this statement, namely: if AB is invertible, then so are A and B.

5. Is the following a subspace of $\mathbf{R}^2$? For each of the three conditions in the definition of a subspace say whether it passes or fails here.

   The set S of all vectors in $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbf{R}^2$ with $x \geq 0$.

6. Determine which of the following sets of vectors are subspaces of $\mathbf{R}^3$ (with explanation).

   (a) The set S of all vectors of the form $[a\ 0\ 0]^T$ where $a \in \mathbf{R}$.

   (b) The set S of all vectors of the form $[a\ 1\ 1]^T$ where $a \in \mathbf{R}$.

   (c) The set S of all vectors of the form $[a\ b\ c]^T$ where $a, b, c \in \mathbf{R}$ and $b = a + c$.

   (d) The set S of all vectors of the form $[a\ b\ c]^T$ where $a, b, c \in \mathbf{R}$ and $b = a + c + 1$.

   (e) The set S of all vectors of the form $[a\ b\ 0]^T$ where $a, b \in \mathbf{R}$.

7. Consider $\vec{u} = \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}$.

   (a) Is $\vec{u}$ in col(A)? Justify.

   (b) Is $\vec{u}^T$ in row(A)? Justify.

   (c) Is $\vec{u}$ in null(A)? Justify.

8. Determine in each part whether the three given vectors span $\mathbf{R}^3$.

   $(a)\quad \vec{v}_2 = \begin{bmatrix} 2 \\ 7 \\ 5 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 9 \\ 6 \end{bmatrix}.$ $\qquad (b)\quad \vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ -4 \\ 5 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}.$

9. In each part determine whether the following four vectors are linearly independent in $\mathbf{R}^4$.

   $(a)\quad \vec{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 4 \\ 3 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 4 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ -7 \end{bmatrix}.$

   $(b)\quad \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -5 \\ 8 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 13 \\ 3 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} -2 \\ 4 \\ -9 \\ -2 \end{bmatrix}..$

10. Show that $\vec{w}$ is in span($\mathcal{B}$) and find the coordinate vector $[\vec{w}]_{\mathcal{B}}$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} \right\}, \qquad \vec{w} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

11. State your outside sources and who you worked with on this assignment. (If you didn't work with anybody and didn't use any other outside sources, clearly write NO OUTSIDE SOURCES.)