# Solutions to Assignment 3

due: On gradescope 11:00PM, Thursday, March 6, 2025

1. Consider the matrices

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 1 & -1 \\ 5 & 1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 1 & 3 \\ -1 & 2 & 3 \\ 6 & 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & 5 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 \\ -1 & 3 \\ 0 & 2 \end{bmatrix} \quad E = \begin{bmatrix} 2 & 0 \\ -4 & 2 \end{bmatrix}$$

Compute the following or explain why it is not possible: a) $A + B$, b) $A - B$, c) $5C$, d) $2D - E$

$$A + B = \begin{bmatrix} 6 & 0 & 5 \\ -1 & 3 & 2 \\ 11 & 2 & 0 \end{bmatrix}$$

$$A - B = \begin{bmatrix} -2 & -2 & -1 \\ 1 & -1 & -4 \\ -1 & 0 & -4 \end{bmatrix}$$

$$5C = \begin{bmatrix} 10 & -5 & 15 \\ 5 & 15 & 25 \end{bmatrix}$$

$D$ and $E$ are different sizes, so $2D - E$ is not defined.

2. Using the matrices from the previous problem, compute the following or explain why it is not possible: a) $CD$, b) $DC$, c) $(CD)E$, d) $C(DE)$.

$$CD = \begin{bmatrix} 3 & 5 \\ -2 & 20 \end{bmatrix}$$

$$DC = \begin{bmatrix} 3 & 2 & 8 \\ 1 & 10 & 12 \\ 4 & 6 & 10 \end{bmatrix}$$

$$(CD)E = C(DE) = \begin{bmatrix} -14 & 10 \\ -84 & 40 \end{bmatrix}$$

The equality $(CD)E = C(DE)$ follows from associativity.

3. (a) Is there a $2 \times 2$ matrix $A$ with $A \neq O_{2,2}$ such that $AA = O_{2,2}$? Justify your answer.
   (b) Give three examples of $2 \times 2$ matrices $A$ with $AA = A$. (Verify your answer on MATLAB.)

a) Yes. For example, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. But $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ also work.

b) $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $O_{2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are obvious. But $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ also work.

4. (a) Give an example of $2 \times 2$ matrices $A$ and $B$ such that $(A + B)(A - B) \neq A^2 - B^2$.
   (b) State a valid formula for multiplying out $(A + B)(A - B)$.
   (c) What condition can you impose on $A$ and $B$ so that $(A + B)(A - B) = A^2 - B^2$?

   (a) There are MANY solutions, but say $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then

$$(A + B)(A - B) = \left( \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \left( \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$$

   However,

$$A^2 - B^2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}^2 - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

(b) Using left and right distributivity we get
$(A+B)(A-B) = A(A-B) + B(A-B) = (AA - AB) + (BA - BB) = A^2 - AB + BA - B^2$.

(c) If $A$ and $B$ commute, that is $AB = BA$, then
$(A+B)(A-B) = A^2 - AB + BA - B^2 = A^2 - AB + AB - B^2 = A^2 + O - B^2 = A^2 - B^2$.

5. Is it possible for an $n \times n$ matrix $A$ to satisfy $A^5 = I_n$ without $A$ being invertible? Explain. No. $I_n = A^5 = A^4 A = AA^4$ implies by definition that $A$ is invertible, and $A^{-1} = A^4$.

6. (a) Can an $n \times n$ matrix $A$ with a row of zeros have an inverse? Explain your reasoning.
   (b) Can an $n \times n$ matrix $A$ with a column of zeros have an inverse? Explain your reasoning.

   (a) There are many ways of seeing that an $n \times n$ matrix with a row of zeros, say in row $i$, can't have an inverse. For example if $B$ is any $n \times n$ matrix, then row $i$ in the product $AB$ must be a zero row (since $(AB)_{ik} = a_{i1}b_{1k} + \cdots + a_{in}b_{nk} = 0b_{1k} + \cdots + 0a_{nk} = 0$ for all $1 \leq k \leq n$). Thus this product can never be the identity matrix (as that has a 1 in each row), and by definition $A$ can have no inverse $B$.

   (b) Again there are many reasons, why an $n \times n$ matrix with a column of zeros, say column $j$, has no inverse. For example performing ANY sequence of row operations on $A$ will always maintain the fact that column $j$ is a zero column. Thus the RREF of $A$ also has a zero column in column $j$, so that it can't be the identity matrix. Thus by Theorem 3.12 $A$ has no inverse.

7. (a) Prove that if $A, B, C$ are matrices such that $C$ is invertible and $AC = BC$, then $A = B$.
   (b) Give an example of nonzero $2 \times 2$ matrices $A, B, C$ such that $AC = BC$, but $A \neq B$.

   (a) If $C$ is invertible and $AC = BC$, then multiplying both sides of this equation by $C^{-1}$ on the right, we get on the left hand side

$$(AC)C^{-1} = A(CC^{-1}) = AI = A,$$

   and on the right hand side

$$(BC)C^{-1} = B(CC^{-1}) = BI = B.$$

   Therefore $A = B$.

   Fix below

   (b) $A = C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (which is not invertible by Problem 6(b)) and $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$. Then $AC = BC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but certainly $A \neq B$.

8. Find the inverse of the following matrix, where $k$ is a fixed non-zero constant: $\begin{bmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 1 & 0 & k & 0 \\ 0 & 1 & 0 & k \end{bmatrix}$

Using the Gaussian elimination algorithm, we get

$$\left[\begin{array}{cccc|cccc} k & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & k & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & k & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_i \leftarrow \frac{1}{k}R_i} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 \\ \frac{1}{k} & 0 & 1 & 0 & 0 & 0 & \frac{1}{k} & 0 \\ 0 & \frac{1}{k} & 0 & 1 & 0 & 0 & 0 & \frac{1}{k} \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - \frac{1}{k}R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{k^2} & 0 & \frac{1}{k} & 0 \\ 0 & \frac{1}{k} & 0 & 1 & 0 & 0 & 0 & \frac{1}{k} \end{array}\right]$$

$$\xrightarrow{R_4 \leftarrow R_4 - \frac{1}{k}R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{k^2} & 0 & \frac{1}{k} & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{k^2} & 0 & \frac{1}{k} \end{array}\right]$$

Therefore the inverse is

$$\begin{bmatrix} \frac{1}{k} & 0 & 0 & 0 \\ 0 & \frac{1}{k} & 0 & 0 \\ -\frac{1}{k^2} & 0 & \frac{1}{k} & 0 \\ 0 & -\frac{1}{k^2} & 0 & \frac{1}{k} \end{bmatrix}.$$

9. Consider the matrices

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 2 & -5 & 2 \\ 7 & -5 & 0 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(a) Show that the equation $A\vec{x} = \vec{x}$ can be rewritten as $(A - I_3)\vec{x} = \vec{0}$ and use this result to solve $A\vec{x} = \vec{x}$ for $\vec{x}$.

(b) Solve $A\vec{x} = -3\vec{x}$.

(a) Adding $-I_3\vec{x}$ to both sides of $A\vec{x} = \vec{x}$ we obtain $(A-I_3)\vec{x} = A\vec{x}-I_3\vec{x} = \vec{x}-\vec{x} = \vec{0}$. Here $B = A-I_3 = \begin{bmatrix} 2 & -3 & 1 \\ 2 & -5 & 2 \\ 7 & -5 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 7 & -5 & -1 \end{bmatrix}$. By MATLAB, $\text{rref}([B \mid \vec{0}]) = \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ and so the solutions are $\vec{x} = \begin{bmatrix} -\frac{1}{2}t \\ -\frac{1}{2}t \\ t \end{bmatrix}$.

(b) Adding $3I_3\vec{x}$ to both sides of $A\vec{x} = 3\vec{x}$ we obtain $(A + 3I_3)\vec{x} = A\vec{x} + 3I_3\vec{x} = 3\vec{x} + 3\vec{x} = \vec{0}$. Here $B = A + 3I_3 = \begin{bmatrix} 2 & -3 & 1 \\ 2 & -5 & 2 \\ 7 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 1 \\ 2 & -2 & 2 \\ 7 & -5 & 3 \end{bmatrix}$. By MATLAB,

$\text{rref}([B \mid \vec{0}]) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and so $\vec{x} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, for any $t \in \mathbf{R}$.

10. Consider $A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

Determine $A^2, A^{-1}, A^{-2}$, and conjecture a general formula for $A^k$ where $k$ is any integer.

$$A^2 = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 36 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}.$$

Similarly for every positive integer $k$, $A^k = \begin{bmatrix} (-6)^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 5^k \end{bmatrix}$.

$$A^{-1} = \begin{bmatrix} 1/(-6) & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

$$A^{-2} = (A^2)^{-1} = \begin{bmatrix} 36 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}^{-1} = \begin{bmatrix} 1/36 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/25 \end{bmatrix}.$$

Similarly for positive integer $k$, $A^{-k} = (A^k)^{-1} = \begin{bmatrix} (-6)^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 5^k \end{bmatrix}^{-1} = \begin{bmatrix} 1/(-6)^k & 0 & 0 \\ 0 & 1/3^k & 0 \\ 0 & 0 & 1/5^k \end{bmatrix}$.

Thus altogether we expect that for every integer $k$, $A^k = \begin{bmatrix} (-6)^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 5^k \end{bmatrix}$.

11. State your outside sources and who you worked with on this assignment. (If you didn't work with anybody and didn't use any other outside sources, clearly write NO OUTSIDE SOURCES.)