Solutions to Assignment 2

due: 11:00PM Thursday, February 13, 2025

1. Find the projection of $\vec{v} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}$ onto $\vec{u} = \begin{bmatrix} 2 \\ 2 \\ \sqrt{2} \end{bmatrix}$.

   $\vec{v} \cdot \vec{u} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ -1/4 \\ -1/2 \end{bmatrix} = 2(1/2) + 2(-1/4) + (-2)(-1/2) = 1 - 1/2 + 1 = \frac{3}{2}$.

   $\vec{u} \cdot \vec{u} = \begin{bmatrix} 1/2 \\ -1/4 \\ -1/2 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ -1/4 \\ -1/2 \end{bmatrix} = (1/2)^2 + (-1/4)^2 + (-1/2)^2 = \frac{1}{4} + \frac{1}{16} + \frac{1}{4} = \frac{9}{16}$

   Thus $\mathrm{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{3/2}{9/16}\vec{u} = \frac{8}{3}\begin{bmatrix} 1/2 \\ -1/4 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -2/3 \\ -4/3 \end{bmatrix}$.

2. Give the vector equation of the line through $P = (3, -2, -1)$ and $Q = (-4, 7, -3)$.

   $\vec{d} = \overrightarrow{PQ} = \begin{bmatrix} -2-1 \\ -1-0 \\ 3-(-1) \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}$, so that $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}$.

3. Give the vector equation of the plane through the points $P = (-1, 2, 1), Q = (1, -1, 2)$ and $R = (2, 1, -1)$.

   $\vec{u} = \overrightarrow{PQ} = \begin{bmatrix} 1-1 \\ 0-1 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and $\vec{v} = \overrightarrow{PR} = \begin{bmatrix} 0-1 \\ 1-1 \\ 1-0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are vectors in this plane. Thus one way of expressing this plane in vector form is $\vec{x} = \overrightarrow{OP} + t\vec{u} + s\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

4. Find the distance from the point $P = (2, 2, 2)$ to the plane $x + y - z = 0$.

   Using the formula from the book, the distance is $\frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}} = \frac{|1\cdot 0-2\cdot 0+2\cdot 0-1|}{\sqrt{1^2+(-2)^2+2^2}} = \frac{|-1|}{\sqrt{9}} = \frac{1}{3}$.

   Otherwise find a point Q on the plane. In fact, for $y = z = 0$ we have $x = 1 + 2y - 2z = 1 - 0 + 0 = 1$, so we can choose $Q = (1, 0, 0)$. With the normal vector $\vec{n} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ and $\vec{v} = \overrightarrow{PQ} = \overrightarrow{OQ} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ we get $\vec{a} = \mathrm{proj}_{\vec{n}}(\vec{v}) = \frac{1}{9}\vec{n}$. Thus the distance is $\|\vec{a}\| = \frac{1}{9}\sqrt{1^2 + (-2)^2 + 2^2} = \frac{1}{3}$.

5. Find the distance between the parallel planes $x + y + z = 1$ and $x + y + z = 3$. (a) is not a linear equation, since products of variables like x_1x_3 should not appear in a linear equation.
   (b) is not a linear equation, since variables should occur only to the first power in a linear equation, unlike x_1^{-2}.
   (c) is a linear equation.

6. In each part, solve the linear system, if possible, and use the result to determine whether the lines represented by the equations in the system have zero, one, or infinitely many points of intersection. If there is a single point of intersection, give its coordinates, and if there are infinitely many, find parametric equations for them.

   (a) $\begin{array}{rcrcr} -2x & - & 3y & = & 5 \\ 8x & + & 12y & = & -25 \end{array}$ (b) $\begin{array}{rcrcr} 3x & - & 7y & = & 2 \\ 9x & - & 21y & = & 6 \end{array}$ (c) $\begin{array}{rcrcr} 3x & - & y & = & 0 \\ 6x & - & y & = & 9 \end{array}$

   You can solve these in any way, like substitution or some form of elimination with or without matrices.

(a)

$$\begin{aligned} 3x - 2y &= 4 \qquad \text{denote this equation as } E_1 \\ 6x - 4y &= 9 \qquad \text{denote this equation as } E_2 \end{aligned}$$

Eliminate x in E_2 by subtracting $2E_1$ from it:

$$\begin{aligned} 6x - 4y - 2(3x - 2y) &= 9 - 2 \times 4 \\ (6-6)x + (4-4)y &= 9 - 8 \\ 0 &= 1 \end{aligned}$$

This contradictory result means the system is inconsistent, so there is no solution. Therefore, the corresponding lines have no point of intersection.

(b)

$$2x - 4y = 1 \qquad \text{denote this equation as } E_1$$
$$4x - 8y = 2 \qquad \text{denote this equation as } E_2$$

Eliminate x in E_2 by subtracting $2E_1$ from it:

$$4x - 8y - 2(2x - 4y) = 2 - 2 \times 1$$
$$(4-4)x + (8-8)y = 2 - 2$$
$$0 = 0$$

The always true statement above $(0 = 0)$ imposes no restriction on x and y. Thus, E_2 is redundant and can be omitted, leaving $2x - 4y = 1$ as the only restriction on x and y. Such a system has infinitely many solutions, and the corresponding lines have infinitely many points of intersection.
Solve x in terms of y : $\quad x = \frac{4y+1}{2} = 2y + \frac{1}{2}$
Assign arbitrary value t to y, and we would have the following solution in parametric form:

$$(x, y) = (2t + 1/2, t) \qquad (t \in \mathbb{R})$$

(c)

$$x - 2y = 0 \qquad \text{denote this equation as } E_1$$
$$x - 4y = 8 \qquad \text{denote this equation as } E_2$$

Eliminate x in E_2 by subtracting E_1 from it:

$$x - 4y - (x - 2y) = 8 - 0$$
$$(1-1)x + (2-4)y = 8$$
$$-2y = 8$$
$$y = -4$$

Replace y by -4 in E_1: $x - 2(-4) = 0$ so $x = -8$
Thus, the system has the unique solution: $x = -8,\ y = -4$; the corresponding lines intersect at the single point $(-8, -4)$.

7. In each part determine whether the matrix is in row-echelon form, reduced row-echelon form, both or neither.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

By definition, a matrix in row echelon form has 2 properties (all-zero rows at bottom, and lower leading entries closer to the right), and a matrix in reduced row echelon form needs to satisfy two additional properties (every leading entry is 1, and a column with a leading 1 has 0s everywhere else). Therefore, a matrix in reduced row echelon form is in both forms.

(a) is in neither forms, for the leading entry in the 3rd row (2) is below, rather than to the right of the leading entry (1) above it.
(b) is clearly in both forms.
(c) is in row echelon form, but not reduced row echelon form due to the 2 in the second row.
(d) is also in both forms, since "all zero rows" are at the bottom, and there is no nonzero row to violate the other properties.
(e) is in neither forms, for the all-zero row is not at the bottom.
(f) is in neither forms: the leading 1 in the 2nd row isn't to the right of the leading 1 in the first row.
(g) is in both forms.

8. In each part suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given row-echelon form. Solve the system.

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 & 4 & 0 & 5 \\ 0 & 1 & -2 & 3 & 1 & 4 \\ 0 & 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(a) Since the augmented matrix has 4 columns, there are 3 variables: x_1, x_2, x_3

$$\begin{aligned} 2 \cdot x_1 - 3 \cdot x_2 + 4 \cdot x_3 &= 7 \\ 1 \cdot x_2 + 2 \cdot x_3 &= 2 \\ 1 \cdot x_3 &= 5 \end{aligned}$$

Using the third equation: $x_3 = 5$
Using the second equation: $x_2 = 2 - 2x_3 = 2 - 10 = -8$
Using the first equation: $x_1 = 7 + 3x_2 - 4x_3 = 7 + 3(-8) - 4(5) = -37$

(b) Since the augmented matrix has 6 columns, there are 5 variables: x_1, x_2, x_3, x_4, x_5

$$\begin{aligned} 1 \cdot x_1 + 7 \cdot x_2 - 2 \cdot x_3 + 0 \cdot x_4 - 8 \cdot x_5 &= -3 \\ 1 \cdot x_3 + 1 \cdot x_4 + 6 \cdot x_5 &= \ \ 5 \\ 1 \cdot x_4 + 3 \cdot x_5 &= \ \ 9 \end{aligned}$$

Solve leading variables in terms of free variables:

Using the third equation: $x_4 = 9 - 3x_5$
Using the second equation: $x_3 = 5 - x_4 - 6x_5 = 5 - (9 - 3x_5) - 6x_5 = -4 - 3x_5$
Using the first equation: $x_1 = -3 - 7x_2 + 2x_3 + 8x_5 = -3 - 7x_2 + 2(-4 - 3x_5) + 8x_5 = -11 - 7x_2 + 2x_5$

Assign arbitrary values to free variables: $x_2 = s,\ x_5 = t$

The solution are all points $(x_1, x_2, x_3, x_4, x_5)$ of the form $(-11-7s+2t, s, -4-3t, 9-3t, t)$ where $s, t \in \mathbb{R}$ are arbitrary scalars.

9. Solve the following system by Gauss-Jordan elimination

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & - & x_3 & = & 0 \\ x_1 & - & 2x_2 & + & 3x_3 & = & -11 \\ 3x_1 & + & 4x_2 & + & 2x_3 & = & 4 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{bmatrix} \xrightarrow{R_3-3R_1} \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$$

$$\xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \xrightarrow{R_3+10R_2} \begin{bmatrix} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} \xrightarrow{(-1/52)R_3} \begin{bmatrix} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1-7R_3} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2+5R_3} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Thus, $x_1 = 3, x_2 = 1, x_3 = 2$ is the unique solution.

10. Let $\mathcal{P}_1$ be the plane defined in question 3. Let $\mathcal{P}_2$ be the plane containing the points $X = (2, 3, -1)$, $Y = (1, -1, 2)$, and $Z = (0, 1, 0)$. Set up an augmented matrix that you would solve to get the intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$. Do not solve! How many solutions do you expect to get, and why?

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix}$$

$$\xrightarrow{R_4-3R_1} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3-3R_2} \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \xrightarrow{R_4-3R_2} \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, if we assign free variables $z = s,\ w = t$, the solution would be $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1+t \\ 2s \\ s \\ t \end{bmatrix}$, where $s, t \in \mathbb{R}$.