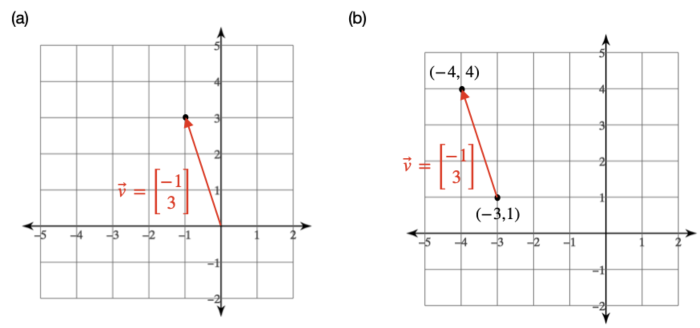
HW 1

Due: Thursday, January 30

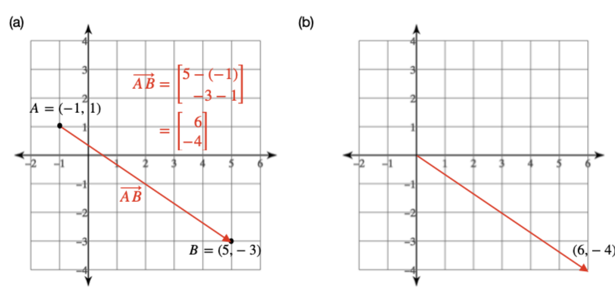
1. Let $\overrightarrow{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.
   (a) Draw the vector in standard position (with its tail at the origin).
   (b) Draw the vector with its tail at $(2, -3)$.

Solution:



2. Given that $A = (1, -1)$ and $B = (4, 2)$, draw $\overrightarrow{AB}$. Compute the vector numerically, and also draw it in standard position.

Solution:



3. Let $\overrightarrow{v}$ be the vector from problem 1, and let $\overrightarrow{u} = \overrightarrow{AB}$ from problem 2. Compute the following algebraically. Also show and explain how to obtain the same result geometrically.
   (a) $\overrightarrow{u} + \overrightarrow{v}$

(b) $-\overrightarrow{v}$

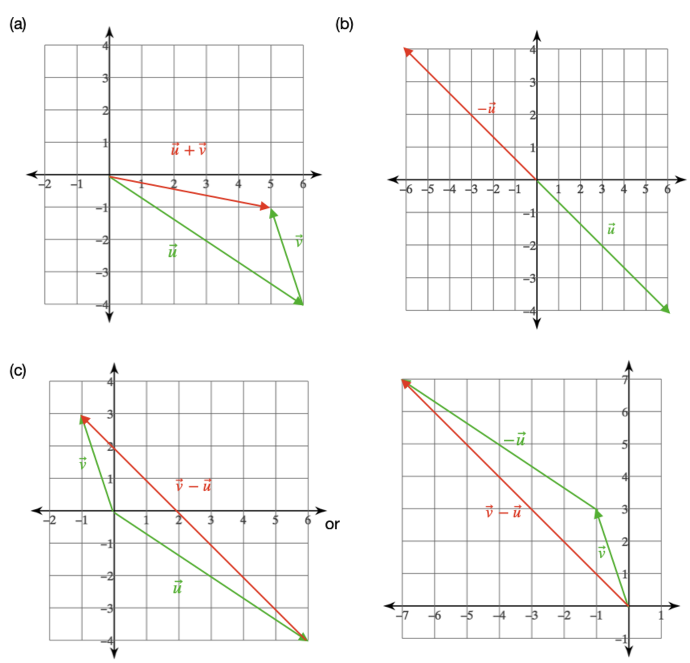
(c) $\overrightarrow{u} - \overrightarrow{v}$

Solution:

(a) $\overrightarrow{u} + \overrightarrow{v} = \begin{bmatrix} 6 \\ -4 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 + (-1) \\ (-4) + 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

(b) $-\overrightarrow{u} = -\begin{bmatrix} 6 \\ -4 \end{bmatrix} = \begin{bmatrix} (-1)6 \\ (-1)(-4) \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$. $-\vec{v}$ has same length, opposite direction as $\overrightarrow{u}$.

(c) $\overrightarrow{v} - \overrightarrow{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 6 \\ -4 \end{bmatrix} = \begin{bmatrix} -1 - 6 \\ 3 - (-4) \end{bmatrix} = \begin{bmatrix} -7 \\ 7 \end{bmatrix}$.



4. If the vector $\overrightarrow{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ has head at $(4, 2, -3)$, where is its tail?

Solution: If $\vec{v} = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}$ is drawn in standard position, then its head is at $(-2, 3, -3)$ and tail $(0, 0, 0)$. To draw it so that its head is at $(-3, -2,\ 1)$ we must shift -1 in x-direction, -5 in y direction, and 4 in z-direction (that is $\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 4 \end{bmatrix}$). So the tail will move to $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 4 \end{bmatrix}$, that is $(-1, -5,\ 4)$.

5. Solve for $\overrightarrow{x}$ in terms of $\overrightarrow{a}$ and $\overrightarrow{b}$:

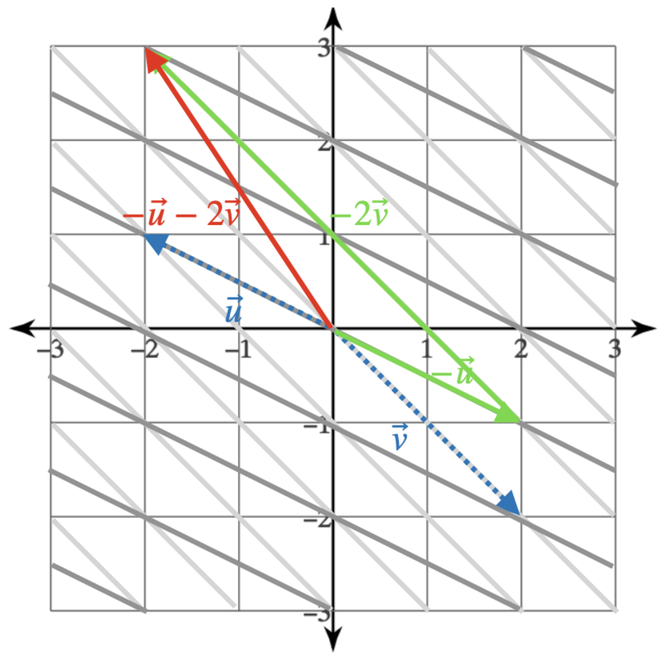
$$\overrightarrow{x} + 2\overrightarrow{a} - \overrightarrow{b} = 3(\overrightarrow{x} + \overrightarrow{a}) - 2(2\overrightarrow{a} - \overrightarrow{b}).$$

Solution:

$$\begin{aligned}
2\vec{x} - \vec{a} - 3\vec{b} &= 2(\vec{a} + \vec{b}) - (\vec{x} - \vec{b}) \\
2\vec{x} - \vec{a} - 3\vec{b} &= 2\vec{a} + 2\vec{b} - \vec{x} + \vec{b} \\
2\vec{x} - \vec{a} - 3\vec{b} &= 2\vec{a} + 2\vec{b} + \vec{b} - \vec{x} \\
2\vec{x} - \vec{a} - 3\vec{b} &= 2\vec{a} + 3\vec{b} - \vec{x} && | + \vec{x} + \vec{a} + 3\vec{b} \\
2\vec{x} + \vec{x} &= 2\vec{a} + 3\vec{b} + \vec{a} + 3\vec{b} \\
3\vec{x} &= 2\vec{a} + \vec{a} + 3\vec{b} + 3\vec{b} \\
3\vec{x} &= 3\vec{a} + 6\vec{b} \\
\vec{x} &= \vec{a} + 2\vec{b}
\end{aligned}$$

6. Draw the coordinate grid corresponding to the vectors $\overrightarrow{u} = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$ and $\overrightarrow{v} = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. Compute $\overrightarrow{w} = 2\overrightarrow{u} + 3\overrightarrow{v}$ and draw $\overrightarrow{w}$ using the grid.

Solution: $\overrightarrow{u} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \overrightarrow{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$. Thus $-\vec{u} - 2\vec{v} = -\begin{bmatrix} -2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \cdot (-2) - 2 \cdot 2 \\ -1 \cdot 1 - 2 \cdot (-2) \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.



7. If $\vec{u} = \left[\begin{smallmatrix} 3 \\ -2 \end{smallmatrix}\right]$ and $\vec{v} = \left[\begin{smallmatrix} 4 \\ 6 \end{smallmatrix}\right]$, what are $\vec{u} \cdot \vec{v}$ and $\|\vec{u}\|$?

Solution: $\vec{u} \cdot \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 1 \cdot 3 + 2 \cdot 2 + 2 \cdot 1 = 3 + 4 + 2 = 9.$

$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{1 \cdot 1 + 2 \cdot 2 + 2 \cdot 2} = \sqrt{9} = 3.$

8. Suppose $\vec{u}, \vec{v}, \vec{w}$ are vectors in $\mathbb{R}^n$, c is a scalar, and $\cdot$ denotes the dot product. Explain why the expressions below don't make sense:
   (a) $\|\vec{u} \cdot \vec{v}\|$
   (b) $\vec{u} \cdot \vec{v} + \vec{w}$
   (c) $\vec{u} \cdot (\vec{v} \cdot \vec{w})$
   (d) $c \cdot (\vec{u} + \vec{v})$

Solution:

(a) $\vec{u} \cdot \vec{v}$ is a number, but we can only take the norm of a vector, so $\|\vec{u} \cdot \vec{v}\|$ makes no sense.

(b) $\vec{u} \cdot \vec{v}$ is a number and $\vec{w}$ a vector, but we can't add a number and a vector, so $\vec{u} \cdot \vec{v} + \vec{w}$ makes no sense.

(c) $\vec{v} \cdot \vec{w}$ is a number, $\vec{u}$ is a vector, so $\vec{u} \cdot (\vec{v} \cdot \vec{w})$ makes no sense, no matter if you interpret the first $\cdot$ as a dot product, scalar product, or multiplication of real numbers.

(d) $\vec{u} + \vec{v}$ is a vector, so $c \cdot (\vec{u} + \vec{v})$ makes no sense since we can't do a dot product of a number and a vector.

9. Prove Theorem 1.2b: if $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ are vectors in $\mathbb{R}^n$, then

$$\overrightarrow{u} \cdot (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} \cdot \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{w}.$$

Solution:

$$\begin{aligned}
\vec{u} \cdot c\vec{v} = \vec{u} \cdot \left(c \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right) &= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix} \\
&= u_1(cv_1) + u_2(cv_2) + \cdots + u_n(cv_n) \\
&= (u_1 c)v_1 + (u_2 c)v_2 + \cdots + (u_n c)v_n \\
&= (cu_1)v_1 + (cu_2)v_2 + \cdots + (cu_n)v_n \\
&= c(u_1 v_1) + c(u_2 v_2) + \cdots + c(u_n v_n) \\
&= c(u_1 v_1 + u_2 v_2 + \cdots + u_n v_n) \\
&= c(\vec{u} + \vec{v})
\end{aligned}$$

10. Prove that if $\overrightarrow{u}, \overrightarrow{v}$ are vectors in $\mathbb{R}^n$, then

$$\|\overrightarrow{u} - \overrightarrow{v}\|^2 = \|\overrightarrow{u}\|^2 - 2\overrightarrow{u} \cdot \overrightarrow{v} + \|\overrightarrow{v}\|^2.$$

Solution:

$$\begin{aligned}
\|\vec{u}-\vec{v}\|^2 &= (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot(\vec{u}-\vec{v}) - \vec{v}\cdot(\vec{u}-\vec{v}) \\
&= \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} \\
&= \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} \\
&= \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2
\end{aligned}$$