Name:

Math 264: Exam 3

You may use MATLAB on the classroom computer. Make sure that MATLAB is the only application open, it is maximized, and there are no commands typed into the window prior to the start of the exam.

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. You may use any MATLAB command covered in the quickstart guide on my webpage. Write "ML" next to any calculation that was done with MATLAB.

You may use any result from the sections covered in the text or from lecture. You may not use the results of homework problem.

Please do not put any work on the back of pages. Use the space on the last page instead.

1. True or false. No justification is required.

   (a) (5 pts) If $\det(A) = 8$ and $\det(B) = 4$, then $\det(AB^{-1}) = 2$.

   True False

   Solution: True:

   $$\det(AB^{-1}) = \det(A)\det(B^{-1}) = \det(A)\det(B)^{-1} = 8 \cdot 4^{-1} = 2.$$

   (b) (5 pts) If 3 is an eigenvalue of A, then the rows of $A - 3I$ are linearly dependent.

   True False

   Solution: True: $A - 3I$ has nonzero nullspace, so by the Fundamental Theorem of Invertible Matrices, the rows are linearly dependent.

   (c) (5 pts) The function $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ x + y - 1 \end{bmatrix}$ is a linear transformation.

   True False

   Solution: False: $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, but for a linear transformation we should get back $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

2. True or false. If true, explain why. If false, give an example to show it is false.

   (a) (10 pts) If A and B are square matrices with the same rref, then they have the same eigenvalues.

   Solution: False: Take $A = I_2$ and $B = 2I_2$. Since B is invertible, its rref is I_2. But it has eigenvalue 2 only, while A has eigenvalue 1 only.

   (b) (10 pts) If A is a 4×4 matrix, then $\det(-A) = -\det(A)$.

   Solution: False: Take $A = I_4$. Then $\det A = 1$, but

   $$\begin{aligned}\det(-A) &= \det \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\ &= (-1)^4 \\ &= 1.\end{aligned}$$

(c) (10 pts) Two eigenvectors of the same eigenvalue must be linearly independent.

Solution: False: take $A = I_2$, $\overrightarrow{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\overrightarrow{v}_2 = \begin{bmatrix}2\\0\end{bmatrix}$. We have $A\overrightarrow{v}_1 = 1 \cdot \overrightarrow{v}_1$ and $A\overrightarrow{v}_2 = 1 \cdot \overrightarrow{v}_2$, so both are eigenvectors with eigenvalue 1. But $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ are parallel, so linearly dependent.

(d) (10 pts) If the eigenvalues of 4×4 matrix are 1 and -1 only, then the matrix is invertible.

Solution: True: it is not invertible if and only if 0 is an eigenvalue, but the Fundamental Theorem of Invertible Matrices.

3. (15 pts) Find the standard matrix for the linear transformation that consists of a reflection about the y-axis, followed by a counterclockwise rotation by $30°$, followed by a reflection about the line $y = x$.

Solution: With T be reflection across the y-axis, $R_{30°}$ the rotation by $\theta = 30°$ counterclockwise, and S be the reflection about the line $y = x$, we see that our transformation is $S \circ R_{30°} \circ T$. We have

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}-x\\y\end{bmatrix},$$

and thus its standard matrix is $\left[\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right]$. Similarly, the standard matrix for S is $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$. Thus the matrix for $S \circ R_{30°} \circ T$ is

$$\begin{aligned}
\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}\cos 30° & -\sin 30°\\ \sin 30° & \cos 30°\end{bmatrix}\begin{bmatrix}-1&0\\0&1\end{bmatrix} &= \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}-1&0\\0&1\end{bmatrix}\\
&= \begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}-1&0\\0&1\end{bmatrix}\\
&= \begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}.
\end{aligned}$$

(Note that this is the same as the matrix for rotation by $240°$.)

4. Suppose that

$$\det\begin{bmatrix}1&2&3\\4&-6&2\\a&b&c\end{bmatrix} = 6.$$

Compute the following, and justify your answer.

(a) (10 pts) $\det\begin{bmatrix}1-2a & 2-2b & 3-2c\\ 4 & -6 & 2\\ a & b & c\end{bmatrix}$

Solution: We obtain this matrix by subtracting twice the third row from the first. This leaves the determinant unchanged. Therefore the determinant of the new matrix is 6.

(b) (10 pts) $\det\begin{bmatrix}a&b&c\\2&-3&1\\1&2&3\end{bmatrix}$

Solution: We get this by first swapping rows 1 and 3, then multiplying the 2nd row by $\frac{1}{2}$. The first row operation multiplies the determinant by -1; the second multiplies it by $\frac{1}{2}$. The result is $(-1)(\frac{1}{2})(6) = -3$.

5. (10 pts) Let

$$\vec{v} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & 1 & 0 & 4 \\ 1 & 4 & -3 & 8 \\ 3 & 4 & 7 & -1 \\ 0 & 2 & 5 & -4 \end{bmatrix}.$$

Show that $\vec{v}$ is an eigenvector of B, and compute its eigenvalue.

Solution: We use MATLAB to compute

$$B\vec{v} = \begin{bmatrix} 15 \\ -10 \\ -5 \\ -5 \end{bmatrix} = 5 \begin{bmatrix} 3 \\ -2 \\ -1 \\ -1 \end{bmatrix}.$$

Therefore it is an eigenvector with eigenvalue 5.

6. (15 pts) Let

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

Show that A is not diagonalizable.

Solution: The characteristic polynomial is

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} -2-\lambda & 1 & 0 \\ 0 & -2-\lambda & 4 \\ 0 & 0 & 3-\lambda \end{bmatrix} \\ &= (-2-\lambda)(-2-\lambda)(3-\lambda) \\ &= -(\lambda+2)^2(\lambda-3). \end{aligned}$$

Setting this equal to 0, we get $\lambda = -2$ or $\lambda = 3$. The algebraic multiplicity of 3 is 1, so the geometric multiplicity must also be 1. The algebraic multiplicity of -2 is 2, so we just need to show that the geometric multiplicity is smaller than 2; in other words, 1.

The geometric multiplicity we want is $\dim E_{-2}$. To get that, we need the null space of $A + 2I$. We have

$$A + 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{bmatrix}.$$

MATLAB tells us the rref is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

There is one free variable, so $\dim E_{-2} = 1$, as desired. Therefore A is not diagonalizable.

7. Let
$$A = \begin{bmatrix} 8 & 9 \\ -6 & -7 \end{bmatrix}.$$

(a) (15 pts) Diagonalize A.

Solution: The characteristic polynomial of A is
$$\det(A - \lambda I) = \det \begin{bmatrix} 8-\lambda & 9 \\ -6 & -7-\lambda \end{bmatrix} = (8-\lambda)(-7-\lambda) + 54 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1).$$
Setting this equal to zero, we get $\lambda = 2$ or $\lambda = -1$.
Now we compute the eigenspaces. For $\lambda = 2$, we want the null space of
$$\begin{bmatrix} 6 & 9 \\ -6 & -9 \end{bmatrix} \text{ which from MATLAB has rref } \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 0 \end{bmatrix}$$
and so the eigenspace is spanned by $[-3/2, 1]$. I'll use $[-3, 2]$ instead to avoid fractions.
For $\lambda = -1$, we want the null space of
$$\begin{bmatrix} 9 & 9 \\ -6 & -6 \end{bmatrix} \text{ which from MATLAB has rref } \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$
so the eigenspace is spanned by $[1, -1]$.
Now applying our diagonalization theorem, we have $A = PDP^{-1}$ where
$$D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } P = \begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}$$
One should also compute P^{-1}, but this is good enough.

(b) (10 pts) If $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, compute $A^{99}\vec{v}$.

Solution: First we solve
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} -3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$
A straightforward computation shows that $c_1 = -3$ and $c_2 = -8$. Thus
$$\begin{aligned} A^{99}\vec{v} &= A^{99}\left(-3\begin{bmatrix} -3 \\ 2 \end{bmatrix} - 8\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \\ &= -3A^{99}\begin{bmatrix} -3 \\ 2 \end{bmatrix} - 8A^{99}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= -3 \cdot 2^{99}\begin{bmatrix} -3 \\ 2 \end{bmatrix} - 8 \cdot (-1)^{99}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 9 \cdot 2^{99} \\ -3 \cdot 2^{100} \end{bmatrix} + \begin{bmatrix} 8 \\ -8 \end{bmatrix} \\ &= \begin{bmatrix} 9 \cdot 2^{99} + 8 \\ -3 \cdot 2^{100} - 8 \end{bmatrix}. \end{aligned}$$

8. (10 pts) Suppose that A is a 3×3 matrix with eigenvectors $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$ having respective

eigenvalues $3, -3$, and 1. Compute A.

Solution: From our diagonalization theorem, we have

$$A = PDP^{-1}$$

where

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } P = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix}.$$

Plugging into MATLAB, we get

$$A = \begin{bmatrix} 5 & -4 & -12 \\ 0 & 3 & 6 \\ \frac{4}{3} & -\frac{8}{3} & -7 \end{bmatrix}.$$