Name:

Math 264: Exam 2

You may use MATLAB on the classroom computer. Make sure that MATLAB is the only application open, and that it is maximized.

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. You may use any MATLAB command covered in the quickstart guide on my webpage. Write "ML" next to any calculation that was done with MATLAB.

You may use any result from the sections covered in the text or from lecture. You may not use the results of homework problem.

Please do not put any work on the back of pages. Use the space on the last page instead.

1. True or false. If true, provide justification. If false, give an example that shows it is false. (Two points for correct answer, four for justification/example.)

   (a) (6 pts) If A and B are matrices such that $AB = O$ and $A \neq O$, then $B = O$.

   True False

   Solution: False: Take $A = B = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. A computation with MATLAB, or by hand, shows that $AB = A^2 = O$.

   (b) (6 pts) If a matrix has more rows than columns, then the dimension of its row space is larger than the dimension of its column space.

   True False

   Solution: False. The dimensions of the row and column space are always the same.

   (c) (6 pts) Every line in $\mathbb{R}^2$ is a subspace.

   True False

   Solution: False. The line must go through $(0, 0)$.

   (d) (6 pts) If A is a square matrix whose rows add up to the zero vector, then A is not invertible.

   True False

   Solution: True. The rows are linearly dependent, and the result follows from the Fundamental Theorem of Invertible Matrices.

2. Determine if each of the following set S is a subspace. Justify your answer.

   (a) (10 pts) The solution set in $\mathbb{R}^3$ of

$$\begin{aligned} 2x + y - z &= 0 \\ x + 2y + z &= 0 \\ x - y - 2z &= 0. \end{aligned}$$

   Solution: Yes: it is a null space, which is a subspace. Specifically, it is the null space of

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & -1 & -2 \end{bmatrix}.$$

(b) (10 pts) The solution set in $\mathbb{R}^3$ of

$$\begin{aligned} 2x + y - z &= 1 \\ x + 2y + z &= -2 \\ x - y - 2z &= 3. \end{aligned}$$

Solution: No: $[0,0,0]$ does not satisfy the first equation (or any of the equations, for that matter), so does not lie in the set. But a subspace must contain the zero vector.

(c) (10 pts) The set of vectors $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ with $x \geq 0$.

Solution: No: notice that $[1,0]$ is in the set since $1 \geq 0$. But $-1[1,0] = [-1,0]$ is not in the set. Therefore the set does not have the scalar multiplication property.

(d) (10 pts) The set of vectors $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ with $y = x^2$.

Solution: No: notice that $[1,1]$ is in the set since $1^2 = 1$. But $2[1,1] = [2,2]$, and $2^2 \neq 2$. Therefore $2[1,1]$ is not in the set, so the set fails the scalar multiplication property.

3. Let A be the matrix

$$\begin{bmatrix} 2 & -4 & 5 & 8 & 5 \\ 1 & -2 & 2 & 3 & 1 \\ 4 & -8 & 3 & 2 & 6 \end{bmatrix}.$$

(a) (10 pts) Find a basis for the row space of A.

Solution: We use MATLAB to find the rref, which is

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The nonzero rows then form a basis: $[1,-2,0,-1,0]$, $[0,0,1,2,0]$, and $[0,0,0,0,1]$.

(b) (10 pts) Find a basis for the column space of A.

Solution: In the rref, the leading entries are in columns 1, 3, and 5. Taking those columns of the original matrix gives the answer:

$$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix}$$

(c) (10 pts) Find a basis for the null space of A.

Solution: From the rref, the free variables are x_2 and x_4. We set $x_2 = s$ and $x_4 = t$. Our equations from the rref are

$$\begin{aligned} x_1 - 2x_2 \quad - x_4 &= 0 \\ x_3 + 2x_4 &= 0 \\ x_5 &= 0. \end{aligned}$$

Substituting s and t and solving yields

$$\begin{aligned} x_1 &= 2s + t \\ x_2 &= s \\ x_3 &= -2t \\ x_4 &= t \\ x_5 &= 0. \end{aligned}$$

Thus our solutions are

$$\begin{bmatrix} 2s+t \\ s \\ -2t \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore our basis is $[2, 1, 0, 0, 0], [1, 0, -2, 1, 0]$.

4. (10 pts) Given that the matrix $\begin{bmatrix} 2 & t \\ 3 & 4 \end{bmatrix}$ is *not* invertible, what is the value of t?

Solution: We want the determinant to be zero. The determinant is $2 \cdot 4 - t \cdot 3 = 8 - 3t$. Setting equal to 0 and solving yields $t = \frac{8}{3}$.

5. (10 pts) Let

$$B = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

Suppose $B^{-1}AC^2 - B = O_{2,2}$. Compute the matrix A.

Solution: We have $B^{-1}AC^2 = B + O = B$. Multiplying both sides on the left by B gives

$$AC^2 = B^2.$$

Multiplying on the right by C^{-2} yields $A = B^2C^{-2}$. Now we plug into MATLAB to get

$$A = \begin{bmatrix} -\frac{1}{4} & 2 \\ -\frac{3}{4} & 2 \end{bmatrix}.$$

6. (10 pts) Find the coordinates of $\begin{bmatrix} 13 \\ 16 \\ -13 \end{bmatrix}$ with respect to the basis $\left(\begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right)$. Express your answer as a column vector.

Solution: We solve the augmented system

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 13 \\ 1 & 3 & -1 & 16 \\ -4 & 5 & 1 & -13 \end{array}\right].$$

Computing the rref in MATLAB yields

$$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Therefore the coefficient vector is $[7, 3, 0]$.

7. (10 pts) Give an example of a matrix D which simultaneously satisfies the following properties:

- the row space of D is 2-dimensional,
- the null space of D is 2-dimensional, and
- $\left[\begin{smallmatrix} 1 \\ 2 \\ 0 \end{smallmatrix}\right]$ is in the column space of D.

Justify that your matrix works.

Solution: By the Rank-Nullity Theorem, D must have 4 columns. From the 3rd bullet, it must have 3 rows. We might as well assume D is in rref; then it should have exactly 2 leading entries. If the vector $[1, 2, 0]$ *is* one of the columns, then it's definitely in the column space! However, $[1, 2, 0]$ cannot appear in a column with a leading entry (since for such a column, exactly one of the entries is 1, and the rest 0). So let's try

$$D = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

And indeed it works: 2 leading entries so row space has dimension 2, 2 free columns so the nullity is 2, and $[1, 2, 0]$ is one of the columns, so is in the column space.