Name:

Math 264: Exam 1

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Calculators are not allowed.

You may use any result from the chapters covered in the text or from lecture. You may not use the results of homework or worksheet problems.

Please do not put any work on the back of pages. Use the space on the last page instead.

1. True or false. Circle your answer. No justification is required.

   (a) (5 pts) If a system of linear equations is consistent, then it has no free variables.

   True False

   Solution: False: $x + y = 0$ is consistent and has 1 free variable.

   (b) (5 pts) In $\mathbb{R}^3$, if a direction vector for a line is the same as one of the direction vectors for a plane, then the line lies in the plane.

   True False

   Solution: False: Take $\vec{x} = [0,0,1] + s[1,0,0] + t[0,1,0]$. This is the plane parallel to the xy-plane but 1 unit up. Then take the line $\vec{x} = [0,0,0] + t[1,0,0]$. All points have z-coordinate 0, so this line does not lie in the plane.

   (c) (5 pts) In $\mathbb{R}^3$, if $\text{proj}_{\vec{w}}\vec{v} = \vec{0}$, then $\vec{v}$ and $\vec{w}$ are orthogonal.

   True False

   Solution: True: the numerator in the projection formula is $\vec{v} \cdot \vec{w}$.

   (d) (5 pts) If $\vec{u}, \vec{v}, \vec{w}$ are three vectors in $\mathbb{R}^n$ with $\vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w}$, then $\vec{u} = \vec{v}$.

   True False

   Solution: False: take $\vec{w} = \vec{0}$ and $\vec{u}, \vec{v}$ anything.

   (e) (5 pts) If c is a scalar and $\vec{v}$ is a vector, then $\|c\vec{v}\| = c\|\vec{v}\|$.

   True False

   Solution: False: $\|-1\vec{v}\| = 1\|\vec{v}\|$.

2. (5 pts) Given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$, define what it means for the list of vectors to be *linearly dependent*.

   Solution: If there are scalars $c_1, \ldots, c_k$ not all 0 for which

   $$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}.$$

3. (10 pts) Find the angle between the vectors $[1,0,1]$ and $[2,2,1]$.

   Solution: If θ is the angle, then

   $$\begin{aligned}\cos\theta &= \frac{[1,0,1]\cdot[2,2,1]}{\|[1,0,1]\|\|[2,2,1]\|}\\ &= \frac{3}{\sqrt{2}\cdot\sqrt{9}}\\ &= \frac{1}{\sqrt{2}}.\end{aligned}$$

   Therefore $\theta = \frac{\pi}{4}$ (or 45°).

4. (10 pts) The solution set of the system of equations

$$\begin{aligned}2x + y - 3z &= -2\\ 3x + 4y - 7z &= -13\\ x - 2y + z &= 9\\ x - 7y + 6z &= 29,\end{aligned}$$

forms a line. Determine the rank of the matrix

$$\begin{bmatrix} 2 & 1 & -3 \\ 3 & 4 & -7 \\ 1 & -2 & 1 \\ 1 & -7 & 6 \end{bmatrix}.$$

Justify your answer.

Solution: The matrix is the coefficient matrix for the system of equations. As a line uses 1 free parameter, there must be 1 free variable, and hence 2 leading variables. Therefore the rank is 2.

5. (10 pts) Solve the following system of equations by Gaussian elimination:

$$\begin{aligned} x + 2y \quad\quad &= 1 \\ 2x + y - 2z &= 0 \\ 3x \quad\; - 4z &= -1. \end{aligned}$$

Solution: We get

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 2 & 1 & -2 & 0 \\ 3 & 0 & -4 & -1 \end{array}\right] \xrightarrow{R2 \leftarrow R2 - 2R1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & -3 & -2 & -2 \\ 3 & 0 & -4 & -1 \end{array}\right]$$

$$\xrightarrow{R3 \leftarrow R3 - 3R1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & -3 & -2 & -2 \\ 0 & -6 & -4 & -4 \end{array}\right]$$

$$\xrightarrow{R3 \leftarrow R3 - 2R2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & -3 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R2 \leftarrow -\frac{1}{3}R2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right].$$

That means z is a free variable; we set $z = t$. The second equation yields $y + \frac{2}{3}z = \frac{2}{3}$, which is the same as $y = \frac{2}{3} - \frac{2}{3}t$. The first equation is $x + 2y = 1$, which is the same as

$$x + 2\left(\frac{2}{3} - \frac{2}{3}t\right) = 1$$

or $x = -\frac{1}{3} + \frac{4}{3}t$. In sum:

$$\begin{aligned} x &= -\frac{1}{3} + \frac{4}{3}t \\ y &= \frac{2}{3} - \frac{2}{3}t \\ z &= t. \end{aligned}$$

6. (10 pts) Prove that if $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w}$ are vectors in $\mathbb{R}^2$, then

$$\overrightarrow{u} \cdot (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} \cdot \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{w}.$$

Solution: Write $\overrightarrow{u} = [x_1, y_1]$, $\overrightarrow{v} = [x_2, y_2]$, and $\overrightarrow{w} = [x_3, y_3]$. Then

$$\begin{aligned}
\overrightarrow{u} \cdot (\overrightarrow{v} + \overrightarrow{w}) &= [x_1, y_1] \cdot ([x_2, y_2] + [x_3, y_3]) \\
&= [x_1, y_1] \cdot [x_2 + x_3, y_2 + y_3] \\
&= x_1(x_2 + x_3) + y_1(y_2 + y_3) \\
&= x_1x_2 + x_1x_3 + y_1y_2 + y_1y_3 \\
&= (x_1x_2 + y_1y_2) + (x_1x_3 + y_1y_3) \\
&= \overrightarrow{u} \cdot \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{w}.
\end{aligned}$$

7. (10 pts) Find a parametric vector equation for the line through the point $(1, 2, 3)$ and perpendicular to the plane with equation $2x - y + z = 5$.

Solution: The plane has normal vector $\vec{n} = [2, -1, 1]$. Since the line is perpendicular to the plane, $\vec{n}$ is parallel to the line. Thus our equation is

$$\overrightarrow{x} = [1, 2, 3] + t[2, -1, 1].$$

8. Let $A = (1, 0, 2)$, $B = (0, 1, -2)$, $C = (1, 1, -1)$, and $D = (4, 2, 1)$.

(a) Compute $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

Solution:

$$\begin{aligned}
\overrightarrow{AB} &= [0, 1, -2] - [1, 0, 2] = [-1, 1, -4] \\
\overrightarrow{CD} &= [4, 2, 1] - [1, 1, -1] = [3, 1, 2].
\end{aligned}$$

(b) Compute $2\overrightarrow{AB} - 3\overrightarrow{CD}$.

Solution:

$$\begin{aligned}
2\overrightarrow{AB} - 3\overrightarrow{CD} &= 2[-1, 1, -4] - 3[3, 1, 2] \\
&= [-2, 2, -8] + [-9, -3, -6] \\
&= [-11, -1, -14].
\end{aligned}$$

(c) Compute $\|\overrightarrow{AB}\|$.

Solution:

$$\begin{aligned}\|\overrightarrow{AB}\| &= \sqrt{\overrightarrow{AB}\cdot\overrightarrow{AB}} \\ &= \sqrt{(-1)^2+1^2+(-4)^2} \\ &= \sqrt{18} \\ &= 3\sqrt{2}.\end{aligned}$$

(d) Compute $\operatorname{proj}_{\overrightarrow{CD}}\overrightarrow{AD}$.

Solution:

$$\begin{aligned}\operatorname{proj}_{\overrightarrow{CD}}\overrightarrow{AD} &= \frac{\overrightarrow{AD}\cdot\overrightarrow{CD}}{\overrightarrow{CD}\cdot\overrightarrow{CD}}\overrightarrow{CD} \\ &= \frac{[3,2,-1]\cdot[3,1,2]}{[3,1,2]\cdot[3,1,2]}[3,1,2] \\ &= \frac{9}{14}[3,1,2].\end{aligned}$$

If you like, this last can be rewritten $[\frac{27}{14},\frac{9}{14},\frac{9}{7}]$; but it isn't necessary.

(e) Find an equation for the plane through A, B, and C. (Your choice of form.)

Solution: Parametric is way easier. We need a point and two nonparallel direction vectors. Any point will do; let's pick A. For our direction vectors, let's use $\overrightarrow{AB} = [-1,1,-4]$ and $\overrightarrow{AC} = [0,1,-3]$. Then our equation is

$$\vec{x} = [1,0,2] + s[-1,1,-4] + t[0,1,-3].$$

9. Determine if $[0,1]$ lies in $\operatorname{Span}([1,2],[3,-1])$.

Solution: We need to determine if the augmented system

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & -1 & 1 \end{array}\right]$$

is consistent. Subtracting twice the first row from the second yields

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -7 & 1 \end{array}\right]$$

which is in echelon form. Since the rank is 2, the system is consistent, so the answer is "yes".