Number Six: What do you want?
Number Two: Information!
Number Six: Whose side are you on?
Number Two: That would be telling. We want information!
Number Six: You won't get it!
Number Two: By hook or by crook, we will!
— Opening sequence of "The Prisoner" (1967–68)

22 More Lower Bounds (November 16)

22.1 What *Are* Lower Bounds?

So far in this class we've been developing algorithms and data structures for solving certain problems and analyzing their time and space complexity.

Let $T_A(X)$ denote the running of algorithm A given input X. Then the worst-case running time of A for inputs of size n is defined as follows:

$$T_A(n) = \max_{|X|=n} \big(T_A(X)\big).$$

The worst-case complexity of a *problem* Π is the worst-case running time of the *fastest* algorithm for solving it:

$$T_\Pi(n) = \min_{A \text{ solves } \Pi} \big(T_A(n)\big) = \min_{A \text{ solves } \Pi} \left(\max_{|X|=n} \big(T_A(X)\big)\right).$$

Now suppose we've shown that the worst-case running time of an algorithm A is $O(f(n))$. Then we immediately have an *upper bound* for the complexity of Π:

$$T_\Pi(n) \le T_A(n) = O(f(n)).$$

The faster our algorithm, the better our upper bound. In other words, when we give a running time for an algorithm, what we're really doing — and what most theoretical computer scientists devote their entire careers doing[1] — is bragging about how *easy* some problem is.

Starting with the previous lecture, we've turned the tables. Instead of bragging about how easy problems are, now we're arguing that certain problems are *hard* by proving *lower bounds* on their complexity. This is a little harder, because it's no longer enough to examine a single algorithm. To show that $T_\Pi(n) = \Omega(f(n))$, we have to prove that *every* algorithm that solves Π has a worst-case running time $\Omega(f(n))$, or equivalently, that *no* algorithm runs in $o(f(n))$ time.

[1] This sometimes leads to long sequences of results that sound like an obscure version of "Name that Tune":

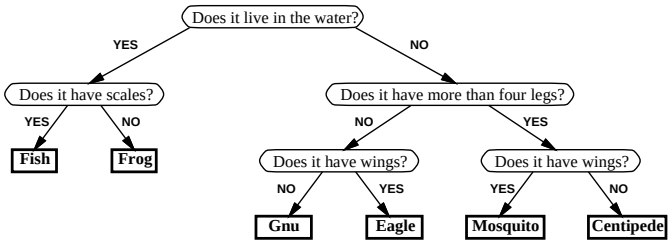
Lennes: "I can triangulate that polygon in $O(n^2)$ time."
Shamos: "I can triangulate that polygon in $O(n \log n)$ time."
Tarjan: "I can triangulate that polygon in $O(n \log\log n)$ time."
Seidel: "I can triangulate that polygon in $O(n \log^* n)$ time."
[Audience gasps.]
Chazelle: "I can triangulate that polygon in $O(n)$ time."
[Audience gasps and applauds.]
"Triangulate that polygon!"

22.2 Decision Trees

Unfortunately, there is no formal definition of the phrase "all algorithms"![2] So when we derive lower bounds, we first have to specify, formally, what an algorithm is and how to measure its running time. This specification is called a *model of computation*.

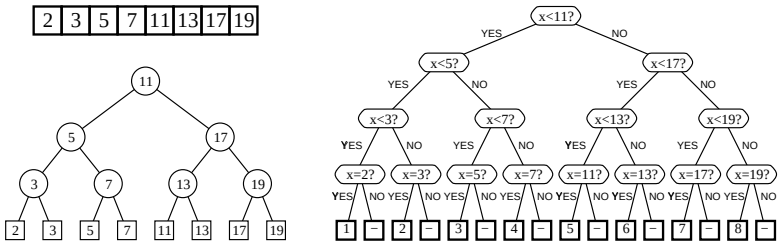
One rather powerful model of computation is *decision trees*. A decision tree is (as the name suggests) a tree. Each internal node in the tree is labeled by a *query*, which is just a question about the input. The edges out of a node correspond to the various answers to the query. Each leaf of the tree is labeled with an *output*. To compute with a decision tree, start at the root and follow a path down to a leaf. At each internal node, the answer to the query tells you which node to visit next. When you reach a leaf, output its label.

For example, the guessing game where one person thinks of an animal and the other person tries to figure it out with a series of yes/no questions can be modeled as a decision tree. Each internal node is labeled with a question and has two edges labeled "yes" and "no". Each leaf is labeled with an animal.



A decision tree to choose one of six animals.

Here's another simple example, called the *dictionary problem*. Let A be a fixed array with n numbers. Suppose want to determine, given a number x, the position of x in the array A, if any. One solution to the dictionary problem is to sort A (remembering every element's original position) and then use binary search. The (implicit) binary search tree can be used almost directly as a decision tree. Each internal node the the *search* tree stores a key k; the corresponding node in the *decision* tree stores the question "Is $x < k$?". Each leaf in the *search* tree stores some value $A[i]$; the corresponding node in the *decision* tree asks "Is $x = A[i]$?" and has two leaf children, one labeled "i" and the other "none".



Left: A binary search tree for the first eight primes.
Right: The corresponding binary decision tree for the dictionary problem ($-$ = "none").

We *define* the running time of a decision tree algorithm for a given input to be the number of queries in the path from the root to the leaf. For example, in the "Guess the animal" tree above,

[2] Complexity-theory purists might argue that "all algorithms' is just a synonym for "all Turing machines". (If you want to know what a Turing machine is, take 375.) In my opinion, this is nonsense. Or it might not be nonsense, but it isn't a particularly *useful* definition. Turing machines are just another model of computation.

$T(\text{frog}) = 2$. Thus, the worst-case running time of the algorithm is just the depth of the tree. This definition ignores other kinds of operations that the algorithm might perform that have nothing to do with the queries. (Even the most efficient binary search problem requires more than one machine instruction per comparison!) But the number of decisions is certainly a *lower bound* on the actual running time, which is good enough to prove a lower bound on the complexity of a problem.

Both of the examples describe *binary* decision trees, where every query has only two answers. We may sometimes want to consider decision trees with higher degree. For example, we might use queries like "Is x greater than, equal to, or less than y?" or "Are these three points in clockwise order, colinear, or in counterclockwise order?" A k-*ary* decision tree is one where every query has (at most) k different answers. **From now on, I will only consider k-ary decision trees where k is a constant.**

22.3 Information Theory

Most lower bounds for decision trees are based on the following simple observation: *the answers to the queries must give you enough information to specify any possible output.* If a problem has N different outputs, then obviously any decision tree must have at least N leaves. (It's possible for several leaves to specify the same output.) Thus, if every query has at most k possible answers, then the depth of the decision tree must be at least $\lceil \log_k N \rceil = \Omega(\log N)$.

Let's apply this to the dictionary problem for a set S of n numbers. Since there are $n + 1$ possible outputs, any decision tree must have at least $n+1$ leaves, and thus any decision tree must have depth at least $\lceil \log_k(n+1) \rceil = \Omega(\log n)$. So the complexity of the dictionary problem, in the decision-tree model of computation, is $\Omega(\log n)$. This matches the upper bound $O(\log n)$ that comes from a perfectly-balanced binary search tree. That means that the standard binary search algorithm, which runs in $O(\log n)$ time, is *optimal*—there is no faster algorithm in this model of computation.

22.4 But wait a second...

We can solve the membership problem in $O(1)$ expected time using hashing. Isn't this inconsistent with the $\Omega(\log n)$ lower bound?

No, it isn't. The reason is that hashing involves a query with more than a constant number of outcomes, specifically "What is the hash value of x?" In fact, if we don't restrict the degree of the decision tree, we can get constant running time even without hashing, by using the obviously unreasonable query "For which index i (if any) is $A[i] = x$?". No, I am *not* cheating — remember that the decision tree model allows us to ask *any* question about the input!

This example illustrates a common theme in proving lower bounds: *choosing the right model of computation is absolutely crucial.* If you choose a model that is too powerful, the problem you're studying may have a completely trivial algorithm. On the other hand, if you consider more restrictive models, the problem may not be solvable at all, in which case any lower bound will be meaningless! (In this class, we'll just tell you the right model of computation to use.)

22.5 Sorting

Now let's consider the *sorting* problem — Given an array of n numbers, arrange them in increasing order. Unfortunately, decision trees don't have any way of describing moving data around, so we have to rephrase the question slightly:

> Given a sequence $\langle x_1, x_2, \ldots, x_n \rangle$ of n distinct numbers, find the permutation π such that $x_{\pi(1)} < x_{\pi(2)} < \cdots < x_{\pi(n)}$.

Now a k-ary decision-tree lower bound is immediate. Since there are $n!$ possible permutations π, any decision tree for sorting must have at least $n!$ leaves, and so must have depth $\Omega(\log(n!))$. To simplify the lower bound, we apply *Stirling's approximation*

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left(1 + \Theta\left(\frac{1}{n}\right)\right) > \left(\frac{n}{e}\right)^n.$$

This gives us the lower bound

$$\lceil \log_k(n!) \rceil > \left\lceil \log_k \left(\frac{n}{e}\right)^n \right\rceil = \lceil n \log_k n - n \log_k e \rceil = \Omega(n \log n).$$

This matches the $O(n \log n)$ upper bound that we get from mergesort, heapsort, or quicksort, so those algorithms are optimal. The decision-tree complexity of sorting is $\Theta(n \log n)$.

Well... we're not quite done. In order to say that those algorithms are optimal, we have to demonstrate that they fit into our model of computation. A few minutes thought will convince you that they can be described as a special type of decision tree called a *comparison* tree, where every query is of the form "Is x_i bigger or smaller than x_j?" These algorithms treat any two input sequences exactly the same way as long as the same comparisons produce exactly the same results. This is a feature of any comparison tree. In other words, *the actual input values don't matter, only their order*. Comparison trees describe almost all sorting algorithms: bubble sort, selection sort, insertion sort, shell sort, quicksort, heapsort, mergesort, and so forth — but *not* radix sort or bucket sort.

22.6 Finding the Maximum

Finally let's consider the *maximum* problem: Given an array X of n numbers, find its largest entry. Unfortunately, there's no hope of proving a lower bound in this formulation, since there are an infinite number of possible answers, so let's rephrase it slightly.

> Given a sequence $\langle x_1, x_2, \ldots, x_n \rangle$ of n distinct numbers, find the index m such that x_m is the largest element in the sequence.

We can get an upper bound of $n - 1$ comparisons in several different ways. The easiest is probably to start at one end of the sequence and do a linear scan, maintaining a current maximum. Intuitively, this seems like the best we can do, but the information-theoretic lower bound is only $\lceil \log_2 n \rceil$.

To prove that we can't do better than $n-1$ comparisons, we use an *adversary argument*. The adversary originally pretends that $x_i = i$ for all i, and answers all comparison queries appropriately. Whenever the adversary reveals that $x_i < x_j$, he marks x_i as an item that the algorithm knows

(or should know) is *not* the maximum element. At most one element x_i is marked after each comparison, and x_n is never marked. If the algorithm stops before doing $n-1$ comparisons, at least one other element $x_k < x_n$ is still unmarked. The adversary can change the value of x_k from k to $n+1$, making x_k the largest element, without being inconsistent with any of the comparisons that the algorithm has performed. In other words, the algorithm cannot tell that the adversary has cheated. However, x_n is the maximum element in the original input, and x_k is the largest element in the modified input, so the algorithm cannot possibly give the correct answer for both inputs. Thus, in order to be correct, any algorithm must perform at least $n-1$ comparisons.

As usual, this adversary strategy makes *no* assumptions about the order in which the algorithm does its comparisons. The adversary forces *any* algorithm in this model of computation[3] to either perform $n-1$ comparisons, or to give the wrong answer for at least one input sequence.

22.7 Finding the Minimum and Maximum

Let's consider the complexity of finding the largest *and* smallest elements in a set of n numbers. More formally:

> Given a sequence $X = \langle x_1, x_2, \ldots, x_n \rangle$ of n distinct numbers, find indices i and j such that $x_i = \min X$ and $x_j = \max X$.

How many comparisons do we need to solve this problem? An upper bound of $2n-3$ is easy: find the minimum in $n-1$ comparisons, and then find the maximum of everything else in $n-2$ comparisons. Similarly, a lower bound of $n-1$ is easy, since any algorithm that finds the min and the max certainly finds the max.

We can improve the upper and the lower bound to $\lceil 3n/2 \rceil - 2$. The upper bound is given by the following algorithm. Compare all $\lfloor n/2 \rfloor$ consecutive pairs of elements x_{2i-1} and x_{2i}, and put the smaller element into a set S and the larger element into a set L. if n is odd, put x_n into both L and S. Then find the smallest element of S and the largest element of L. The total number of comparisons is at most

$$\underbrace{\left\lfloor \frac{n}{2} \right\rfloor}_{\text{build } S \text{ and } L} + \underbrace{\left\lceil \frac{n}{2} \right\rceil - 1}_{\text{compute min } S} + \underbrace{\left\lceil \frac{n}{2} \right\rceil - 1}_{\text{compute max } L} = \left\lceil \frac{3n}{2} \right\rceil - 2.$$

For the lower bound, we use an adversary argument. The adversary marks each element $+$ if it might be the maximum element, and $-$ if it might be the minimum element. Initially, the adversary puts both marks $+$ and $-$ on every element. If the algorithm compares two double-marked elements, then the adversary declares one smaller, removes the $+$ mark from the smaller element, and removes the $-$ mark from the larger one. In every other case, the adversary can answer so that at most one mark needs to be removed. For example, if the algorithm compares a double marked element against one labeled $-$, the adversary says the one labeled $-$ is smaller and removes the $-$ mark from the other. If the algorithm compares to $+$'s, the adversary must unmark one of the two.

[3] Actually, the $n-1$ lower bound for finding the maximum holds in a much powerful model called *algebraic* decision trees, which are binary trees where every query is a comparison between two polynomial functions of the input values, such as "Is $x_1^2 - 3x_2x_3 + x_4^{17}$ bigger or smaller than $5 + x_1x_3^5x_5^2 - 2x_7^{42}$?"

Initially, there are $2n$ marks. At the end, in order to be correct, exactly one item must be marked $+$ and exactly one other must be marked $-$, since the adversary can make any $+$ the maximum and any $-$ the minimum. Thus, the algorithm must force the adversary to remove $2n-2$ marks. At most $\lfloor n/2 \rfloor$ comparisons remove two marks; every other comparison removes at most one mark. Thus, the adversary strategy forces any algorithm to perform at least $2n-2-\lfloor n/2 \rfloor = \lceil 3n/2 \rceil - 2$ comparisons.

22.8 Finding the Median

Finally let's consider the *median* problem: Given an array X of n numbers, find its $n/2$th largest entry. (I'll assume that n is even to eliminate pesky floors and ceilings.) More formally:

> Given a sequence $\langle x_1, x_2, \ldots, x_n \rangle$ of n distinct numbers, find the index m such that x_m is the $n/2$th largest element in the sequence.

To prove a lower bound for this problem, we can use a combination of information theory and two adversary arguments. We use one adversary argument to prove the following simple lemma:

Lemma: Any comparison tree that correctly finds the median element also identifies the elements smaller than the median and larger than the median.

Proof: Suppose we reach a leaf of a decision tree that chooses the median element x_m, and there is still some element x_i that isn't known to be larger or smaller than x_m. In other words, we cannot decide based on the comparisons that we've already performed whether $x_i < x_m$ or $x_i > x_m$. Then in particular no element is known to lie between x_i and x_m. This means that there must be an input that is consistent with the comparisons we've performed, in which x_i and x_m are adjacent in sorted order. But then we can swap x_i and x_m, without changing the result of any comparison, and obtain a different consistent input in which x_i is the median, not x_m. Our decision tree gives the wrong answer for this "swapped" input. □

This lemma lets us rephrase the median-finding problem yet again.

> Given a sequence $X = \langle x_1, x_2, \ldots, x_n \rangle$ of n distinct numbers, find the indices of its $n/2-1$ largest elements L and its $n/2$th largest element x_m.

Now suppose a "little birdie" tells us the set L of elements larger than the median. This information fixes the outcomes of certain comparisons — any item in L is bigger than any element not in L — and so we can "prune" those comparisons from the comparison tree. The pruned tree finds the largest element of $X \setminus L$ (the median of X), and thus must have depth at least $n/2-1$. In fact, an adversary argument similar to last lecture's implies that *every* leaf in the pruned tree must have depth at least $n/2-1$, so the pruned tree has at least $2^{n/2-1}$ leaves.

There are $\binom{n}{n/2-1} \approx 2^n/\sqrt{n/2}$ possible choices for the set L. Every leaf in the original comparison tree is also a leaf in exactly one of the $\binom{n}{n/2-1}$ pruned trees, so the original comparison tree must have at least $\binom{n}{n/2-1}2^{n/2-1} \approx 2^{3n/2}/\sqrt{n/2}$ leaves. Thus, any comparison tree that finds the median must have depth at least

$$\left\lceil \frac{n}{2} - 1 + \lg \binom{n}{n/2-1} \right\rceil = \frac{3n}{2} - O(\log n).$$

A more complicated adversary argument (also involving pruning the tree with little birdies) improves this lower bound to $2n - o(n)$.

A similar argument implies that at least $n - k + \lceil \lg \binom{n}{k-1} \rceil$ comparisons are required to find the kth largest element in an n-element set.