CSC C78F 2000
Assignment 5

due: Monday, December 4, 2000

1. The *complete graph* K_n is the graph on n vertices in which every pair of vertices is joined by an edge.

   (a) Explain how BFS can be used to give an algorithm that checks in time $O(m+n)$ if a graph G with n vertices and m edges is connected.

   (b) Explain why the running time of this algorithm is $\Theta(n^2)$ when the input graph is K_n.

   (c) Modify BFS to obtain an algorithm which checks in $O(m+n)$ time if G is connected, but whose running time for K_n is only $\Theta(n)$.

   (d) Find a *disconnected graph* on n vertices for which the running time of the algorithm from (c) is $\Theta(n^2)$, or explain why no such graph exists.

   (e) Find a *connected graph* on n vertices for which the running time of the algorithm from (c) is $\Theta(n^2)$, or explain why no such graph exists.

2. Two trees are edge-disjoint if there is no edge appearing in both of them.

   (a) Let G be a graph with 2 edge-disjoint spanning trees. What is the least number of vertices, n', that G can have? Give an example of a graph on n' vertices which has 2 edge-disjoint spanning trees.

   (b) For general $n \geq n'$, describe a graph on n vertices and a weight function such that the graph has two edge-disjoint minimum spanning trees. Explain why your construction is correct.

   (c) Prove that if a graph has fewer than $2k$ vertices, then it cannot have k spanning trees such that every pair of them is edge-disjoint, when $k \geq 3$.

   (d) Prove that if there is a graph on n vertices that has k pairwise edge-disjoint minimum spanning trees, then there is also a graph on $n+1$ vertices which has k pairwise edge-disjoint minimum spanning trees.

3. We are given a directed graph $G = (V, E)$ that models a communication network. Each edge $(u, v) \in E$ has an associated real number $r(u, v)$ in the interval $[0, 1]$ that represents its *reliability*. In other words, $r(u, v)$ is the probability that the channel from u to v will not fail. We will assume that these probabilities are independent, so that the probability that none of the channels along a path fails is the product of the individual reliabilities.

   Find an efficient algorithm which given vertices u and v, finds a most reliable path from u to v (that is, a path that is least likely to fail). Justify why your algorithm is correct and state its running time.

4. Execute Dijkstra's, Prim's and Kruskal's algorithms on the graph with vertex set $\{0, \ldots 5\}$ and adjacency matrix given below. Use vertex 0 as a source. Break ties between edges by picking the edge whose endpoints give the smallest sum. For example the sum of the endpoints in the edge (1,3) is 1+3=4.

$$Adj = \begin{pmatrix} 0 & 8 & 5 & 9 & 6 & 3 \\ 8 & 0 & 2 & 2 & 5 & 2 \\ 5 & 2 & 0 & 3 & 1 & 7 \\ 9 & 2 & 3 & 0 & 1 & 9 \\ 6 & 5 & 1 & 1 & 0 & 9 \\ 3 & 2 & 7 & 9 & 9 & 0 \end{pmatrix}$$

5. Let G be a graph in which no two edges have the same weight. Prove that G has exactly one minimum weight spanning tree.