Extra Example 2

Linear Algebra (Spring 2005, Prof. Aitken)

Suppose our vector space is $V = \mathbb{R}^2$ and we have two different ordered basis. Suppose that the first ordered basis is

$$v_1 = (3, 1), \quad v_2 = (1, 2)$$

and the second ordered basis is

$$v_1' = (1, 1), \quad v_2' = (0, 1).$$

What is the change of basis matrix from v_1, v_2 to v_1', v_2'? We will calculate this by a clever trick. First, let B_1 be the change of basis matrix from v_1, v_2 to e_1, e_2. Since $v_1 = 3e_1 + 1e_2$ and $v_2 = 1e_1 + 2e_2$ the answer is

$$B_1 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}.$$

Next, let B_2 be the change of basis matrix from e_1, e_2 to v_1', v_2'. By LA21, problem 8, we can first find the change of basis matrix from v_1', v_2' to e_1, e_2 and invert it. Since $v_1' = e_1 + e_2$ and $v_2' = e_2$,

$$B_2^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

This matrix is easy to invert:

$$B_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

By LA21, problem 7, the change of basis matrix from v_1, v_2 to v_1', v_2' is

$$B = B_2 B_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}.$$

(Check this: is $v_1 = 3v_1' - 2v_2'$? is $v_2 = v_1' + v_2'$?)

Now let us illustrate the formula from Problem 5 of LA21:

$$\mathrm{Col}_{(v_i')}(v) = B \cdot \mathrm{Col}_{(v_i)}(v)$$

where (v_i), (v_i'), and B are as above. Let us try it out on $v = 7v_1 + 2v_2$. What is v in terms of the second basis. Observe

$$\mathrm{Col}_{(v_i')}(v) = B \cdot \mathrm{Col}_{(v_i)}(v) = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 23 \\ -12 \end{bmatrix}.$$

So $v = 23v_1' - 12v_2'$.

(To check this, we can go back to the standard coordinates:

$$7v_1 + 2v_2 = 7(3, 1) + 2(1, 2) = (23, 11)$$

and

$$23v_1' - 12v_2' = 23(1, 1) - 12(0, 1) = (23, 11).$$

So the formula works!)