

Extra Example 1

Linear Algebra (Spring 2005, Prof. Aitken)

April 23, 2005

Consider $f(x, y, z) = (x + y, x + z)$ as a function $\mathbb{R}^3 \mapsto \mathbb{R}^2$. For the standard basis, the matrix is

$$\text{Mat}(f) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Let $v_1 = (1, 1, 1)$, $v_2 = (1, 0, 1)$ and $v_3 = (2, -1, 0)$. Let $w_1 = (2, 3)$ and $w_2 = (3, 2)$. How do we find the matrix relative to these ordered bases? One way is to use change of basis matrices, another is the direct approach. Let us try the direct approach. Observe that

$$f(v_1) = f(1, 1, 1) = (2, 2) = \frac{2}{5}(2, 3) + \frac{2}{5}(3, 2) = \frac{2}{5}w_1 + \frac{2}{5}w_2,$$

$$f(v_2) = f(1, 0, 1) = (1, 2) = \frac{4}{5}(2, 3) - \frac{1}{5}(3, 2) = \frac{4}{5}w_1 + \left(-\frac{1}{5}\right)w_2,$$

$$f(v_3) = f(2, -1, 0) = (1, 2) = \frac{4}{5}(2, 3) - \frac{1}{5}(3, 2) = \frac{4}{5}w_1 + \left(-\frac{1}{5}\right)w_2.$$

So

$$\text{Mat}_{(v_i), (w_i)}(f) = \begin{bmatrix} \frac{2}{5} & \frac{4}{5} & \frac{4}{5} \\ \frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 & 4 & 4 \\ 2 & -1 & -1 \end{bmatrix}.$$

Now let us illustrate problem 9 on LA20. We want to illustrate

$$\text{Col}_{(w_i)}(f(v)) = \text{Mat}_{(v_i), (w_i)}(f) \cdot \text{Col}_{(v_i)}(v)$$

for some vector v and with f as before. Let us try out the vector $v = 2v_1 - v_2 + 3v_3$ which in standard coordinates is $(7, -1, 1)$ (but that is not important). The column vector for v relative to the ordered basis v_1, v_2, v_3 is

$$\text{Col}_{(v_i)}(v) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Thus

$$\text{Col}_{(w_i)}(f(v)) = \text{Mat}_{(v_i), (w_i)}(f) \cdot \text{Col}_{(v_i)}(v) = \begin{bmatrix} \frac{2}{5} & \frac{4}{5} & \frac{4}{5} \\ \frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{12}{5} \\ \frac{2}{5} \end{bmatrix}.$$

In other words,

$$f(v) = \frac{12}{5}w_1 + \frac{2}{5}w_2.$$

(Of course, going back to standard coordinates,

$$\frac{12}{5}w_1 + \frac{2}{5}w_2 = \frac{12}{5}(2, 3) + \frac{2}{5}(3, 2) = (6, 8),$$

and a direct calculation of $f(v) = f(7, -1, 1)$ also gives $(6, 8)$.)