LA 9

Linear Algebra (Spring 2005)

Problems 1-6: Matrices. Let R be a commutative ring, often a field.

Definition. Let $M_{m,n}(R)$ be the set of m by n matrices with entries in R. (m rows and n columns).

1. Show that $M_{m,n}(R)$ can be identified with R^{mn}. Conclude that $M_{m,n}(R)$ can be made into an R-module. Describe its addition and scalar multiplication. So if F is a field, then $M_{m,n}(F)$ is a vector space.

2. Observe that $M_{m,n}(R)$ is isomorphic to R^{mn}. So if F is a field then $M_{m,n}(F)$ has dimension mn. Describe a basis.

Definition. Let $f : R^n \to R^m$ be a linear map. Then $\mathrm{Mat}(f)$ is defined to be the m by n matrix formed by setting the jth column equal to $f(\mathbf{e}_j)$ where $f(\mathbf{e}_j)$ is written vertically.

3. Find $\mathrm{Mat}(f)$ for the following linear maps. (i) $(a,b,c) \mapsto (2a+c, 3a-2b+c)$, (ii) $(a,b,c) \mapsto (b,a,c)$, (iii) $(a,b) \mapsto (a,b,0)$, (iv) $(a,b,c) \mapsto (a,a,a)$, (v) $(a,b) \mapsto (a+2b, a-b)$, (vi) $(a,b,c) \mapsto (a,b,c)$.

4. Suppose that $f : R^n \to R^m$ is linear, and that $\mathrm{Mat}(f) = [c_{ij}]$ where c_{ij} is the entry in the ith row and jth column. Show that the c_{ij} are uniquely determined by the following n equations (where $1 \le j \le n$):

$$f(\mathbf{e}_j) = \sum_{i=1}^{m} c_{ij}\mathbf{e}_i$$

(The $\mathbf{e}_j$ is a basis vector in R^n, but each $\mathbf{e}_i$ is a basis vector in R^m.)

5. Suppose that $f : R^n \to R^m$ is linear, and that $\mathrm{Mat}(f) = [c_{ij}]$. Show that $(b_1, \ldots, b_m) = f((a_1, \ldots, a_n))$ can be computed by the formula

$$b_i = \sum_{j=1}^{n} c_{ij}a_j.$$

Hint: write $(a_1, \ldots, a_n)$ as $\sum_{j=1}^{n} a_j e_j$, and apply f. Use Problem 4.

6. Conclude that if $f : R^n \to R^m$ is a linear map, then one can compute $f(v)$ for any $v \in R^n$ just by knowing $\mathrm{Mat}(f)$. In other words, the matrix of a linear map gives all the information needed to reconstruct the linear map.

Problems 7-9: Hom spaces. Let R be a commutative ring.

Definition. Let V and W be two vector spaces (or modules) with the same scalar field (or ring). Then $\mathrm{Hom}(V, W)$ is the set of homomorphisms $V \to W$.

7. Let V and W be two vector spaces (or modules) with the same scalar field (or ring). Show that $\mathrm{Hom}(V, W)$ can be made into a vector space (or module).

8. Show that $f \mapsto \mathrm{Mat}(f)$ defines a homomorphism $\mathrm{Hom}(R^n, R^m) \to M_{m,n}(R)$. Hint: use Problem 4 to determine matrix coefficients.

9. Show that the above homomorphism $\mathrm{Hom}(R^n, R^m) \to M_{m,n}(R)$ is injective.

Problems 10-12: Freedom and matrices. We will see that in R^n you have the freedom to send $\mathbf{e}_i$ wherever you like in a linear map $R^n \to V$. For this reason, modules isomorphic to R^n are called *free modules*.

10. Let V be a vector space (or module) with scalar field (or ring) equal to R. Show that if you choose n vectors $v_1, \ldots, v_n$, then there is a unique homomorphism $f : R^n \to V$ sending $\mathbf{e}_i$ to v_i, and this map has formula $f((a_1, \ldots, a_n)) = \sum_{i=1}^{n} a_i v_i$.

11. Show that the above homomorphism $\mathrm{Hom}(R^n, R^m) \to M_{m,n}(R)$ is an isomorphism. Thus *every matrix is the matrix of a unique linear map.* (Assume here that R is commutative.)

12. Show that $\mathrm{Hom}(R^n, R^m)$ has dimension mn. (Assume here that R is commutative.)