

Definition. If $f : V \rightarrow V'$ is a linear map between vector spaces (or modules), then the *kernel of f* or the *null-space of f* is the set $\{v \in V \mid f(v) = 0\}$.

Problems 1–4: Null-spaces (or kernels). Let $f : V \rightarrow V'$ be a linear map between vector spaces (or modules).

1. Show that the kernel of f is a subspace of V .
2. Show f is injective (one-to-one) if and only if the kernel of f is equal to the zero subspace.
3. Show that if u_1, \dots, u_n is a linearly independent sequence of vectors of V_1 , and if f is injective, then $f(u_1), \dots, f(u_n)$ is a linearly independent sequence of vectors of V_2 . (Optional: does this work for infinite families?)
4. Give a counter-example to the above when f is not injective. (Hint look at $F^3 \rightarrow F^2$ defined by the rule $(x_1, x_2, x_3) \mapsto (x_1, x_2)$).

Definition. If $f : V \rightarrow V'$ is a linear map between vector spaces (or modules), then the *image of f* or the *range of f* is the set W of vectors in V' of the form $f(v)$ with $v \in V$.

Problems 5–7: Image spaces (or ranges). Let $f : V \rightarrow V'$ be a linear map between vector spaces (or modules).

5. Show that the image of f is a subspace of V' . Observe that f is surjective (onto) if and only if the image of f is equal to all of V' .
6. Show that if u_1, \dots, u_n span V_1 , and if f is surjective, then $f(u_1), \dots, f(u_n)$ spans V_2 . (Optional: does this work for infinite spanning sets?)
7. Give a counter-example to the above when f is not surjective. (Hint look at $F^2 \rightarrow F^3$ defined by the rule $(x_1, x_2) \mapsto (x_1, x_2, 0)$).

Problems 8–11: Isomorphisms. In mathematics, an *isomorphism* is a homomorphism that has an inverse homomorphism. In algebra, this is usually equivalent to requiring that the homomorphism be bijective (one-to-one and onto). In topology, bijective is not enough. We will show that in linear algebra it is enough.

Definition. If $f : A \rightarrow B$ and $g : B \rightarrow A$ are functions such that $f \circ g = \text{id}_B$ and $g \circ f = \text{id}_A$, then we say that f and g are *inverse functions*.

8. (Review?) Show that a function f has an inverse function if and only if f is injective (one-to-one) and surjective (onto). Functions that are both injective and surjective are *bijective*.

Definition. If $f : V \rightarrow V'$ is a homomorphism that has an inverse function $f^{-1} : V' \rightarrow V$, and if the inverse function f^{-1} is also a homomorphism, then f is called an *isomorphism*. If there is an isomorphism $V \rightarrow V'$, then V and V' are said to be *isomorphic*.

9. Show that all isomorphisms are bijective homomorphisms.
10. Show that all bijective homomorphisms are isomorphisms, proving the following theorem. (Hint: when showing $f^{-1}(w_1 + w_2) = f^{-1}(w_1) + f^{-1}(w_2)$, for example, let $u_1 = f^{-1}(w_1)$ and $u_2 = f^{-1}(w_2)$ which implies $w_1 = f(u_1)$ and $w_2 = f(u_2)$. Work from the left hand side to the right hand side using these substitutions.)

Theorem. A function $f : V \rightarrow V'$ between vector spaces (or R -modules) is an isomorphism if and only if (i) it is a homomorphism, (ii) it is injective (one-to-one), and (iii) it is surjective (onto).

11. Show that if u_1, \dots, u_n is a basis for V_1 and if $f : V_1 \rightarrow V_2$ is an isomorphism, then $f(u_1), \dots, f(u_n)$ is a basis for V_2 . Conclude that if V_1 is a finite dimensional vector space of dimension n , and if V_1 is isomorphic to V_2 , then V_2 is a finite dimensional vector space of the same dimension n . (Optional: does this work for infinite basis? Does this work for modules?)