LA 7

March 1, 2005

Linear Algebra (Spring 2005)

In mathematics, a *homomorphism* is a function that preserves structure. If a homomorphism is invertible, then it is called an *isomorphism*. Modules and vector space have four parts to their structures (D1-D4), so a homomorphism is a map preserving all four. However, we will show that preserving D1 and D2 automatically implies that D3 and D4 are preserved.

Definition (Version 1). Let V_1 and V_2 be vector spaces (or modules). Assume that V_1 and V_2 have the same scalar field (or ring). Then a *homomorphism* is a function $f : V_1 \to V_2$ that preserves all four part of the structure: (i) $f(u+v) = f(u) + f(v)$ for all $u, v \in V_1$. (ii) $f(cu) = cf(u)$ for all $u \in V_1$ and scalars c. (iii) $f(0) = 0$. (iv) $f(-u) = -f(u)$ for all $u \in V_1$.

Definition (Version 2). Let V_1 and V_2 be vector spaces (or modules). Assume that V_1 and V_2 have the same scalar field (or ring). Then a *homomorphism* is a function $f : V_1 \to V_2$ that satisfies the following two requirements: (i) $f(u+v) = f(u) + f(v)$ for all $u, v \in V_1$. (ii) $f(cu) = cf(u)$ for all $u \in V_1$ and scalars c.

Problems 1–7: Linear maps. In linear algebra, homomorphisms between vector spaces (and sometimes homomorphisms between modules) are called *linear maps* or *linear transformations*.

1. Show that both the above definitions are equivalent, so either could be used as the official definition.

2. Let F be a field. The simplest linear maps $F^n \to F^m$ involve projection or permutation of coordinates. Show that the following are linear maps: (i) $f : F^2 \to F^2$ defined by $(a, b) \mapsto (b, a)$. (ii) $f : F^3 \to F^2$ defined by $(a, b, c) \mapsto (b, c)$, (iii) $f : F^n \to F^1$ defined by $(a_1, \ldots, a_n) \mapsto (a_i)$. The last is called the *ith projection map*. Do your proofs generalize to modules: where F is replaced by a ring R?

3. Let F be a field (or a ring). Let $f : F^3 \to F^3$ be defined by $(a, b, c) \mapsto (b - 2a, 3a + 2b, a - c)$. Show that f is a linear map. (Optional: how would you represent it by a matrix?)

4. Suppose that $f : V \to V'$ and $g : V' \to V''$ are linear maps between vector spaces (or R-modules). Show that the composition $g \circ f$ is a homomorphism. What about $f \circ g$? What has to be true for the composition of two linear maps to be defined?

5. Is composition between vector spaces (over the same scalar field) associative, $f \circ (g \circ h) = (f \circ g) \circ h$, whenever the triple compositions are defined? Why? Also note that the identity map $\mathrm{id}_V : V \to V$ is a homomorphism, and that $\mathrm{id}_V \circ f = f$ and $g \circ \mathrm{id}_V = g$ where f and g are homomorphisms such that the left hand side of the equations are defined (when are the left hand sides of the equations defined?)

Note: between this problem and the previous problem, we have shown that the collection of vector spaces (over the same scalar field) and linear maps between them form a *category*. Categories are very important in mathematics. In this case the vector spaces are the *objects* of the category and the linear maps are the *arrows* (or *morphisms*). Arrows in categories have some of the same properties as elements of a groups except sometimes composition is not defined (also not all arrows have inverses). Observe that everything in this problem generalizes to R-modules. (We will not have cause to use category theory in this course, but I mention it due to its importance).

6. Show that a function $f : A \to A'$ between $\mathbb{Z}$-modules is a module homomorphism (linear map) if and only if it is a group homomorphism. Explain the following statement: "the category of $\mathbb{Z}$-modules is equivalent to the category of abelian groups". (In this and the next problem, you may consult an abstract algebra textbook.)

7. Find a non-trivial $\mathbb{Z}$-module homomorphism $\mathbb{Z}_6 \to \mathbb{Z}_2$. Find a non-trivial $\mathbb{Z}$-module homomorphism $\mathbb{Z}^3 \to \mathbb{Z}_3$. Are there any non-trivial $\mathbb{Z}$-module homomorphism $\mathbb{Z}_6 \to \mathbb{Z}_4$? What about $\mathbb{Z}_6 \to \mathbb{Z}_7$? (*Non-trivial* means that the image is not just $\{0\}$).