# LA 6

February 22, 2005

**Linear Algebra (Spring 2005)**

*Problems 1–2: The Basis Size Theorem (our first main theorem).*

**1**. Show that in the replacement lemma of LA5, that it is impossible for $m > n$. Hint: suppose $n < m$ and use the replacement lemma for $j = \min(n, m)$.

**2**. Prove the following lemmas and theorem.

**Lemma.** *Suppose that there is a set of $m$ linearly independent vectors in a vector space $V$. Then any spanning set must have at least $m$ elements.*

**Lemma.** *Suppose that there is a set of $n$ vectors that span a vector space $V$. Then any linearly independent set is finite with at most $n$ vectors.*

**Theorem (Basis Size Theorem).** *Suppose that $V$ is a finite dimensional vector space. Then every basis is finite, and any two bases have the same number of elements.*

*Problems 3–11: Dimension.*

**Definition.** Let $V$ be a finite dimensional vector space. Then we know that $V$ has a finite basis, and all bases are finite of the same size. The size of a basis of $V$ is called the *dimension* of $V$. The zero subspace 0 is considered to have dimension 0 and basis the empty set (by convention 0 is considered a linear combination of the vectors in the empty set: a sum with no terms is considered to be 0).

**3**. Prove the following.

**Theorem.** *Suppose that $V$ is a vector space of dimension $n$. Then any spanning set has at least $n$ elements, and any spanning set with exactly $n$ elements is a basis. In addition, any linearly independent set of vectors has at most $n$ elements, and any set of linearly independent vectors with exactly $n$ elements is a basis.*

**4**. Show that if $V$ is a vector space of dimension $n$ and $W$ is a proper subspace, then $W$ is a finite dimensional vector space of dimension strictly less than $n$. Hint: Choose as many linearly independent vectors from $W$ as possible.

**5**. Show that if $V$ is a vector space of dimension $n$, and $0 \leq m \leq n$, then there is a subspace of $V$ of dimension $m$. If $m = 0$ or $m = n$ show uniqueness.

**6**. Show that $F^\omega$ is not a finite-dimensional vector space. Hint: find an infinite set of linearly independent vectors and use the above theorem. (Does your infinite set of linearly independent vectors span???)

**7**. Show that if a vector space is infinite dimensional if and only if it has an infinite set of linearly independent vectors. Show that if a vector space $V$ has an infinite dimensional subspace $W$, then $V$ is itself infinite dimensional.

**8**. Show that the set of differentiable functions $\mathbb{R} \to \mathbb{R}$ is infinite dimensional. (Hint: they include $\mathbb{R}[x]$).

**9**. Show that $F^n$ has dimension $n$ (over the scalar field $F$). Show that the dimension of $\mathbb{C}$ over $\mathbb{R}$ is two. What is the dimension of $\mathbb{C}$ over $\mathbb{C}$? What is the dimension of $\mathbb{C}^2$ over $\mathbb{C}$? What is the dimension of $\mathbb{C}^2$ over $\mathbb{R}$?

**10**. Show that every hyperplane of $F^n$ (containing $\mathbf{0}$) is a proper subspace of $F^n$. Conclude that hyperplanes have dimension at most $n-1$. Hint: show that some $\mathbf{e}_i$ not in the hyperplane. (See the new LA3).

**11**. Show that every hyperplane of $F^n$ containing $\mathbf{0}$ has dimension $n-1$. Hint: find $n-1$ linearly independent vectors. It is convenient to reduce to the case where the equation defining the hyperplane can be written as

$$a_1x_1 + \ldots + a_{n-1}x_{n-1} - x_n = 0.$$

*Problems 12–13: Miscellaneous.*

**12**. Show that if $v_1, v_2, v_3$ is a basis of a vector space $V$, and assume that the characteristic of the scalar field is not 2. Show that $v_1 + v_2, v_1 - v_2, v_1 + v_3$ is a basis as well. What about if the characteristic of $F$ is 2?

**13**. For practice: let $V$ be the set of differentiable functions on $\mathbb{R}$. Show that the set of periodic functions with period $2\pi$ is a subspace. Show that the set of even functions and the set of odd functions are subspaces. Without solving it, show that the set of solutions to the differential equation $y' + 7y = 0$ is a subspace. (Now solve it and tell me the dimension).