LA 5

February 13, 2005

Linear Algebra (Spring 2005)

Definition. Above we defined linear dependency and independence for *sets* of vectors. One can define these ideas for *finite sequences* of vectors as well. If $\mathbf{u}_1, \ldots, \mathbf{u}_n$ is a finite sequence of vectors in a vector space (or module) V, then a *linear dependency* is an equation of the form $\sum_{i=1}^{n} a_i \mathbf{u}_i = \mathbf{0}$. A linear dependency is *trivial* if all $a_i = 0$. The finite sequence is said to be *linearly independent* if there are no non-trivial linear dependencies. In fact, linear dependency and independence can be defined for possibly infinite indexed families of vectors as well. The definitions are straightforward, except $a_i = 0$ is required for all but a finite number of i in the index set I.

1. Show that a finite sequence of vectors is linearly independent if and only if (i) it has no redundancies, and (ii) the set of elements on the list are linearly independent in the sense of LA4.

Definition. Let V be a vector space (or a module). A *basis* of V is a linearly independent subset of V which spans V. A finite basis arranged in a finite sequence is called an *ordered* basis.

Problems 2–5: Bases in vector spaces and modules.

2. Show that in a vector space, a set is a basis if and only if it is a minimal spanning set. Give a counter-example for $\mathbb{Z}$-modules. Hint: this question is one that you have already done.

Conclude that every finite spanning set has a subset which is a basis. Conclude also that every finite dimensional vector space has a basis. The module analogue of a finite dimensional vector space is a finitely generated module. Give an example of a finitely generated $\mathbb{Z}$-modules with no basis.

3. A *maximal linearly independent subset* of a vector space V is a linearly independent subset S which cannot be extended to a larger linearly independent subset of V. Show that S is a basis of V if and only if it is a maximal linearly independent subset. Give a counter-example for modules.

Definition. Let R be a ring or a field. In R^n, we write $(1, 0, \ldots, 0)$ as $\mathbf{e}_1$, we write $(0, 1, 0, \ldots, 0)$ as $\mathbf{e}_2$, and in general $\mathbf{e}_i$ is the n-tuple with 1 in the ith position but 0 in all the other positions.

4. Show that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is a basis for R^n. We call this a *canonical basis*. (So some modules have a basis). Find a basis for the set of polynomials of degree at most n in $R[x]$ (where R is a commutative ring).

5. Show that $\mathbf{u}_1, \ldots, \mathbf{u}_n$ is an ordered basis of a vector space (or module) V if and only if the following holds: every $\mathbf{v} \in V$ can be written uniquely in the form $\sum_1^n a_i \mathbf{u}_i$.

Problems 6–9: The Replacement Lemma.

6. To show that two sets of vectors have the same span, just show that every vector in one set is a linear combinations of vectors from the other set. Why does this work?

Definition. Let $\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{v}$ be elements of a vector space V (or module V). We say that $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \ldots, \mathbf{u}_n$ *with essential involvement of* $\mathbf{u}_i$ if

$$a_1 \mathbf{u}_1 + \ldots + a_n \mathbf{u}_n = \mathbf{v}$$

for some scalars $a_1, \ldots, a_n$ with $a_i \neq 0$.

7. Suppose V is a vector space, and $\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{v} \in V$ are such that $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_n$ with essential involvement of $\mathbf{u}_i$. Show that when you replace $\mathbf{u}_i$ by $\mathbf{v}$ you get the same span: $\langle \mathbf{u}_1, \ldots, \mathbf{u}_n \rangle = \langle \mathbf{u}_1, \ldots, \mathbf{u}_{i-1}, \mathbf{v}, \mathbf{u}_{i+1}, \ldots, \mathbf{u}_n \rangle$. (If V is a module, one inclusion can be proved, but what goes wrong in the other direction? Can you tweak the definition of *essential involvement* so that equality holds for modules?).

Lemma (Replacement lemma). *Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be a set of m linearly independent vector in V, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a set of n vectors which span V. Then, for each $0 \leq j \leq \min(m, n)$, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ can be reordered (if necessary) so that $\mathbf{u}_1, \ldots, \mathbf{u}_j, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n$ span V. In other words, you can replace as many of the given spanning vectors as you want by the given linearly independent vectors.*

8. The base case $j = 0$ follows from our assumptions. So assume the result for a particular $j < \min(m, n)$. Show that $\mathbf{u}_{j+1}$ is linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_j, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n$ with essential involvement of $\mathbf{v}_k$ for some $k > j$. Now reorder the $\mathbf{v}_1, \ldots, \mathbf{v}_n$ by switching $\mathbf{v}_k$ and $\mathbf{v}_{j+1}$ if necessary.

9. (Continuation). Conclude that $\mathbf{u}_{j+1}$ is linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_j, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n$ with essential involvement of $\mathbf{v}_{j+1}$. Finish the proof of the lemma.