

Linear Algebra (Spring 2005)

The definition of module is almost the same as the definition of vector space. The only difference is that modules have scalars in a ring, while vector spaces have scalars in a field. Thus every vector space is a module. However, the simple change in possible scalars between vector spaces and modules has serious ramifications: modules can be much less well-behaved than vector spaces.

Definition. If R is a ring, then an R -module is a set M equipped with the following data:

- D1.** A function $M \times M \rightarrow M$ called *addition*, usually written additively.
- D2.** A function $R \times M \rightarrow M$ called *scalar multiplication*, usually written multiplicatively.
- D3.** A distinguished element $\mathbf{0} \in M$ called the *zero element*.
- D4.** A function $M \rightarrow M$ called the *additive inverse function*, denoted $\mathbf{u} \mapsto -\mathbf{u}$.

In addition, M is required to satisfy the following axioms:

- A1.** Associativity: Addition is associative. Scalar multiplication is associative.
- A2.** Commutivity: Addition is commutative.
- A3.** Identity (both types).
- A4.** Inverse: $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ for all $\mathbf{u} \in M$.
- A5.** Distributivity (both types).

Problems 1–5: Modules.

1. Which definitions, examples, theorems and properties from the handouts LA2 and LA3 extend to modules? (Most do, but there are few important exceptions). More specifically, in LA2 investigate numbers 2 and 4, number 5 (just consider rings containing rings), number 6 (but only if the scalar ring is commutative). For LA3 investigate number 1, number 2 (careful here), number 3 (use the term *submodule* instead of subspace), numbers 4, 5, 6, 7, 8, and 9, number 10 (use the term *finitely generated* and *infinitely generated* instead of *finite dimensional* or *infinite dimensional*, and assume R is commutative for polynomials), number 11 (careful here), and number 12.
2. Show that every module is an abelian group (if you forget some of the structure). Conversely, show that if A is an abelian group, then there is a canonical way to define scalar multiplication in such a way that A is a \mathbb{Z} -module. Thus the theory of \mathbb{Z} -modules is equivalent to the study of abelian groups.
3. Show that \mathbb{Z}_n is a \mathbb{Z} -module. Show that $\{1\}$ is a minimal spanning set of \mathbb{Z}_n . Show that, for \mathbb{Z}_6 , the set $\{2, 3\}$ is also a minimal spanning set. Conclude that not every minimal spanning set has the same number of elements. (In contrast, we will show that for vector spaces, the finite minimal spanning sets all have the same number of elements).
4. Show that if A is an abelian group with the property that every element has order dividing n then A is a \mathbb{Z}_n -module. Thus if every non-zero element has order equal to the prime p , then A is a vector space. Show that every abelian group of order 24 is an \mathbb{Z}_{24} -module. Conclude that a set can sometimes be a R -module for several different R (with the same addition).
5. Show that a ring R is itself an R -module (this is a special case of the fact that R^n is a R -module). Show that the submodules of R are exactly the ideals of R . Thus the study of ideals of a ring is equivalent to the study of submodules.

Problems 6–9: Linear independence.

Definition. Let S be a set of elements of a vector space or module V . A *linear dependency* for S is an equation $a_1\mathbf{u}_1 + \dots + a_n\mathbf{u}_n = \mathbf{0}$ where $\mathbf{u}_1, \dots, \mathbf{u}_n$ are distinct elements of S . If each $a_i = 0$, then we call the linear dependency *trivial*. If S has no non-trivial linear dependencies, then we say that S is *linearly independent*. Thus, to show that S is linearly independent, one needs to show that every linear dependency is trivial.

6. Here is another example of the difference between vector spaces and modules. Show that every singleton set $\{\mathbf{u}\}$ with $\mathbf{u} \neq \mathbf{0}$ is linearly independent if V is a vector space. Give a counter-example for \mathbb{Z} -modules.
7. Suppose that \mathbf{u} and \mathbf{v} are distinct non-zero vectors in a vector space. Show that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent if and only if there is no scalar a such that $\mathbf{v} = a\mathbf{u}$. Give a counter-example for \mathbb{Z} -modules.
8. Let V be a vector space. Show that every minimal spanning set S is linearly independent. Hint: see LA3 number 11. Give a counter-example for \mathbb{Z} -modules.
9. Show that a subset of linearly independent vectors is linearly independent (in vector spaces or modules).